Fundamentals and Applications of Perturbation Methods in Fluid Dynamics

Theory and Exercises - JMBC Course - 2018

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Singularity is almost invariably a clue
(Sherlock Holmes, The Boscombe Valley Mystery)

Contents

Chapter 1

Mathematical Modelling and Perturbation Methods

Mathematical modelling is an art. It is the art of portraying a real, often physical, problem mathematically, by sorting out the whole spectrum of effects that play or may play a role, and then making a judicious selection by including what is relevant and excluding what is too small. This selection is what we call a *model* or *theory*. Models and theories, applicable in a certain situation, are not *isolated islands of knowledge* provided with a logical flag, labelling it *valid* or *invalid*. A model is never unique, because it depends on the type, quality and accuracy of answers we are aiming for, and of course the means (time, money, numerical power, mathematical skills) that we have available.

Normally, when the problem is rich enough, this spectrum of effects does not simply consist of two classes *important* and *unimportant*, but is a smoothly distributed hierarchy varying from *essential* effects via *relevant* and *rather relevant* to *unimportant* and *absolutely irrelevant* effects. As a result, in practically any model there will be effects that are small but not small enough to be excluded. We can ignore their smallness, and just assume that all effects that constitute our model are equally important. This is the usual approach when the problem is simple enough for analysis or a brute force numerical simulation.

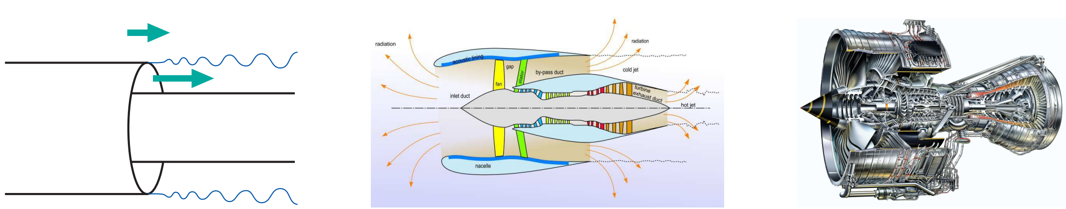


Figure 1.1: Concept of hierarchy (turbofan engine)

There are situations, however, where it could be wise to utilise the smallness of these small but important effects, but in such a way, that we simplify the problem without reducing the quality of the model. Usually, an otherwise intractable problem becomes solvable and (most importantly) we gain great insight in the problem.

Perturbation methods do this in a systematic manner by using the sharp fillet knife of mathematics in general, and asymptotic analysis in particular. From this perspective, perturbation methods are ways of modelling with other means and are therefore much more important for the understanding and analysis of practical problems than they're usually credited with. David Crighton [14] called "*Asymptotics - an indispensable complement to thought, computation and experiment in applied mathematical modelling*".

Examples are numerous: simplified geometries reducing the spatial dimension, small amplitudes allowing linearization, low velocities and long time scales allowing incompressible description, small relative viscosity allowing inviscid models, zero or infinite lengths rather than finite lengths, etc.

The question is: how can we use this gradual transition between models of different level. Of course, when a certain aspect or effect, previously absent from our model, is included in our model, the change is abrupt and big: usually the corresponding equations are more complex and more difficult to solve. This is, however, only true if we are merely interested in exact or numerically *exact* solutions. But an exact solution of an approximate model is not better than an approximate solution of an exact model.

$$x^2 = 4 + 10^{-6}x^5$$
$$x^2 = 4$$

Figure 1.2: Compare "exact" and approximate models.

So there is absolutely no reason to demand the solution to be more exact than the corresponding model. If we accept approximate solutions, based on the inherent small or large modelling parameters, we do have the possibilities to gradually increase the complexity of a model, and study small but significant effects in the most efficient way.

The methods utilizing systematically this approach are called *perturbations methods*. Usually, a distinction is made between regular and singular perturbations. A (loose definition of a) regular perturbation is one in which the solutions of perturbed and unperturbed problem are everywhere close to each other.

We will find many applications of this philosophy in continuous mechanics (fluid mechanics, elasticity), and indeed many methods arose as a natural tool to understand certain underlying physical phenomena. We will consider here four methods relevant in continuous mechanics: (1) the method of slow variation and (2) the method of Lindstedt-Poincaré as examples of regular perturbation methods; then (3) the method of matched asymptotic expansions and (4) the method of multiple scales (with as a special case the WKB method) as examples of singular perturbation methods. In (1) the typical length scale in one direction is much greater than in the others, while in (2) the relevant time scale is unknown and part of the problem. In (3) several approximations, coupled but valid in spatially distinct regions, are solved in parallel. Method (4) relates to problems in which several length scales act in the same direction, for example a wave propagating through a slowly varying environment.

In order to quantify the used small effect in the model, we will always introduce a small positive dimensionless parameter ε. Its physical meaning depends on the problem, but it is always the ratio between two inherent length scales, time scales, or other characteristic problem quantities.

Chapter 2

Modelling and Scaling

2.1 Theory

2.1.1 What is a model? Some philosophical considerations.

Mathematics has, historically, its major sources of inspiration in applications. It is just the unexpected question from practice that forces one to go off the beaten track. Also it is usually easier to portray properties of a mathematical abstraction with a concrete example at hand. Therefore, it is safe to say that most mathematics is applied, applicable or emerges from applications.

Before mathematics can be applied to a real problem, the problem must be described mathematically. We need a mathematical representation of its primitive elements and their relations, and the problem must be formulated in equations and formulas, to render it amenable to formal manipulation and to clarify the inherent structure. This is called mathematical modelling. An informal definition could be:

> *Describing a real-world problem in a mathematical way by what is called a* model*, such that it becomes possible to deploy mathematical tools for its solution. The model should be based on first principles and elementary relations and it should be accurate enough, such that it has reasonable claims to predict both quantitative and qualitative aspects of the original problem. The accuracy of the description should be limited, in order to make the model not unnecessary complex.*

This is evidently a very loose definition. Apart from the question what is meant with: a problem being described in a mathematical way, there is the confusing paradox that we only know the precision of our model, if we can compare it with a better model, but this better model is exactly what we try to avoid as it is usually unnecessarily complex! In general we do not know a problem and its accompanying model well enough to be absolutely sure that the sought description is both consistent, complete and sufficiently accurate for the purpose, ànd not too formidable for any treatment. A model is, therefore, to a certain extent a vague concept. Nevertheless, modelling plays a key rôle in applied mathematics, since mathematics cannot be applied to any real world problem without the intermediate steps of modelling. Therefore, a more structured approach is necessary, which is the aim of the present chapter.

Some people define *modelling* as the process of translating a real-world problem into mathematical terms. We will not do so, as this definition is too wide to include the subtle aspects of "limited precision" (to be discussed below). Therefore we will introduce the word *mathematising*, defined as the process of translating a real-world problem into mathematical terms. It is a translation in the sense that we translate from the inaccurate, verbose "everyday" language to the language of mathematics. For example, the geometrical presence and evolution of objects in space and time may be described parametrically in a suitable coordinate system. Any properties or fields that are expected to play a rôle may be formulated by functions in time and space, explicitly or implicitly, for example as a differential equation.

Mathematising is an elementary but not trivial step. In fact, it forms probably the single most important step in the progress of science. It requires the distinction, naming, and exact specification of the essential relevant elementary objects and their interrelations, where mathematics acts as a language in which the problem is described. If theory is available for the mathematical problem obtained this way, the problem considered may be subjected to the strict logic of mathematics, and reasoning in this language will transcend over the limited and inaccurate ordinary language. Mathematising is therefore, apart from providing the link between the mathematical world and the real world, also important for science in general.

A very important point to note is the fact that such a mathematised formulation is *always* at some level simplified. The earth can be modelled by a point or a sphere in astronomical applications, or by an infinite half-space or modelled not at all in problems of human scale. Based on the level of simplification, sophistication or accuracy, we can associate an inherent hierarchy to the set of possible descriptions. A model may be too crude, but also it may be too refined. It is too crude if it just doesn't describe the problem considered, or if the numbers it produces are not accurate enough to be acceptable. It is too refined if it includes irrelevant effects that make the problem untreatable, or make the model so complicated that important relations or trends remain hidden.

The ultimate goal for mathematising a problem is a deeper understanding and a more profound analysis and solution of the problem. Usually, a more refined problem translation is more accurate but also more complicated and more difficult – if not impossible! – to analyse and solve than a simpler one. Therefore, not every mathematical translation is a good one. We will call a good mathematical translation a *model* or *mathematical model* if it is *lean* or *thrifty* in the sense, that it describes our problem quantitatively or qualitatively in a suitable or required accuracy with a *minimal* number of essentially different parameters and variables. (We say "essentially different", in view of a reduction that is always possible by writing the problem in dimensionless form. See Buckingham's Theorem below.) Again, this definition is rather subjective, as it greatly depends on the context of the problem considered and our knowledge and resources. So there will rarely be one "best" model. At the same time, it shows that modelling, even if relying significantly on intuition, is part of the mathematical analysis.

2.1.2 Types of models

We will distinguish the following three classes of models.

- **Systematic models.**

  Other possible names are *asymptotic models* or *reducing models*, and it is the most important type for us here. The starting point is to use available complete models, which are adequate, but over-complete in so far that effects are included which are irrelevant, uninteresting, or negligibly small, making the mathematical problem unnecessarily complex. By using available additional information (order of magnitude of the parameters) assumptions can be made which minimize in a systematic way the *over-complete* model into a *good* model by taking a parameter that is already large or small to its asymptotic limit: small parameters are taken zero, large parameters become infinite, an almost symmetry becomes a full symmetry.

  Examples of systematic models are found in particular in the well-established fields of continuum physics (fluid mechanics, elasticity). An ordinary flow is usually described by a model which is reduced from the full, *i.e.* compressible and viscous, Navier-Stokes equations.

  An example is the convection-diffusion problem described by the "complete" model

  $$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \alpha \nabla^2 T,$$

  which is difficult to solve, but may be reduced to the much simpler

  $$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = 0$$

  if we have reasons to believe that diffusion term $\alpha \nabla^2 T$ is small compared to convection.

  Another example is the (again difficult) nonlinear pendulum equation

  $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L} \sin\theta,$$

  which may be reduced to the much simpler linear equation

  $$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L}\theta,$$

  if we know or conjecture that angle θ is small and $\sin\theta \simeq \theta$.

- **Constructing models**

  Another possible name is *building block models*. Here we build our problem description step by step from low to high, from simple to more complex, by adding effects and elements lumped together in building blocks, until the required accuracy or adequacy is obtained. This type of model is usually the first if a new scientific discipline is explored.

  An example is the 1D Euler-Bernoulli model of a flexible bar with small displacements and where the bending moment is assumed to be a linear function of the radius of curvature.

  $$EI\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + Q + m_0\frac{\partial^2 y}{\partial t^2} = 0.$$

- **Canonical models.**

  Another possible name is *characteristic models* or *quintessential models*. Here an existing model is further reduced to describe only the essence of a certain aspect of the problem. These models are particularly important if the mathematical analysis of a model from one of the other categories is lacking available theory. The development of such theory is usually hindered by too much irrelevant details. These models are useful for the understanding, but usually far away from the original full problem setting and therefore not suitable for direct industrial application.

  An example is Burgers' equation, originally formulated as an "unphysically" reduced version of the Navier-Stokes equations in order to study certain fundamental effects,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

Note that an asymptotic model may start as a building-block model, which is only found at a later stage to be too comprehensive. Similarly, a canonical model may reduce from an asymptotic model if the latter appears to contain a particular, not yet understood effect, which should be investigated in isolation before any progress with the original model can be made.

The type of model which is most relevant in the context of asymptotic techniques, is the *asymptotic* or *systematic* model. In the following we will explain this further.

2.1.3 Perturbation methods: the continuation of modelling by other means

We have seen above that a real-world problem described by a *systematic* model, is essentially described by a hierarchy of systematic models, where a higher level model is more comprehensive and more accurate than one from a lower level. Now suppose that we have a fairly good model, describing the dominating phenomena in good order of magnitude. And suppose that we are interested in improving on this model by adding some previously ignored aspects or effects. In general, this implies a very abrupt change in our model. The equations are more complex and more difficult to solve. As an illustration, consider the simple "model" $x^2 = a^2$, and the more complete "model" $x^2 + \varepsilon x^5 = a^2$. The first one can be solved easily analytically, the second one with much more effort only numerically. So it seems that the relation between solution and model is not continuous in the problem parameters. Whatever small ε we take, from a transparent and exact solution of the simple model at $\varepsilon = 0$, we abruptly face a far more complicated solution of a model that is just a little bit better. This is a pity, because certain type of useful information (parametric dependencies, trends) become increasingly more difficult to dig out of the more complicated solution of the complex model. This discontinuity of models in the parameter ε may therefore be an argument to retain the simpler model.

The (complexity of the) model is, however, only discontinuous if we are merely interested in exact or numerically "exact" solutions (for example for reasons of benchmarking or validation of solution methods). This is not always the case. As far as our modelling objectives are concerned, we have to keep in mind that also the improved model is only a next step in the modelling hierarchy and not exact in any absolute sense. So there is no reason to require the solution to be more exact than the corresponding model, as **an exact solution of an approximate model is not better than an approximate solution of an exact model**. Moreover, the *type* of information that analytical solutions may provide (functional relationships, *etc.*) is sometimes so important that numerical accuracy may be worthwhile to sacrifice.

Let us go back to our “fairly good”, improved model. The effects we added are relatively small. Otherwise, the previous lower level model was not fairly good as we assumed, but just completely wrong. Usually, this smallness is quantified by small dimensionless parameters (see below) occurring in the equations and (or) boundary conditions. This is the generic situation. The transition from a lower-level to a higher-level theory is characterized by the appearance of one or more modelling parameters, which are (when made dimensionless) small or large, and yield in the limit a simpler description. Examples are infinitely large or small geometries with circular or spherical symmetry that reduce the number of spatial dimensions, small amplitudes allowing linearization, low velocities and long time scales in flow problems allowing incompressible description, small relative viscosity allowing inviscid models, *etc*. In fact, in any practical problem it is really the rule rather than the exception that dimensionless numbers are either small or large.

If we accept approximate solutions, where the approximation is based on the inherently small or large modelling parameters, we do have the possibility to gradually increase the complexity of a model, and study small but significant effects in the most efficient way. The methods utilizing this approach systematically are called “perturbation methods”. The approximation constructed is almost always an asymptotic approximation, *i.e.* where the error reduces with the small or large parameter.

Usually, a distinction is made between *regular* and *singular* perturbations. A (loose definition of a) regular perturbation problem is where the approximate problem is everywhere close to the unperturbed problem. This, however, depends of course on the domain of interest and, as we will see, on the choice of coordinates. If a problem is regular without any need for other than trivial reformulations, the construction of an asymptotic solution is straightforward. In fact, it forms the usual strategy in modelling when terms are linearised or effects are neglected. The more interesting perturbation problems are those where this straightforward approach fails.

We will consider here four methods relevant in the presented modelling problems. The first two are examples of regular perturbation methods, but only after a suitable coordinate transformation. The other two methods are of singular perturbation type, because there is no coordinate transformation possible that renders the problem into one of regular type.

The first method is called the Method of Slow Variation, where the typical axial length scale is much greater than the transverse length scale. The second one is the Lindstedt-Poincaré Method or the method of strained coordinates, for periodic processes. Here, the intrinsic time scale ($\sim$ the period of the solution) is unknown and has to be found. The third one is the Method of Matched Asymptotic Expansions (MAE). To render the problem into one of regular type, different scalings are necessary in spatially distinct regions (boundary layers). The fourth method considered here is the Method of Multiple Scales and may be considered as a combination of the method of slow variation and the method of strained coordinates, as now several (long, short, shorter) length scales occur in parallel. This cannot be repaired by a single coordinate transformation. Therefore, the problem is temporarily reformulated into a higher dimensional problem by taking the various length scales apart. Then the problem is regular again, and can be solved. A refinement of this method is the WKB Method, where the coordinate transformation of the fast variable becomes itself slowly varying.

2.1.4 Nondimensionalisation

The *General Invariance Principle* (whose universality extends far beyond physics) states that the *laws of equilibrium and motion can be expressed through equations valid for all observers*. Hence, the chosen units of a formulation should not be relevant. As a result we have

2.1.4.1 Buckingham's Π-Theorem:

> **Theorem:** If a physical problem is described by n variables and parameters in r dimensions, the number of dimensionless groups is at least[1] $n - r$.

Exactly $n - r$ if all r dimensions play a role. More than $n - r$ if some dimensions are redundant, or occur in the same combination. In that case r is effectively smaller.

Note: mol, rad or dB do *not* count, because they are dimensionless units.

A way to see this theorem intuitively is as follows.
From the problem variables, parameters, and their combinations we can construct time, length, etc. scales. They follow from the problem and are therefore called inherent (length, time) scales. For example, from a velocity V and a length L we have a time L/V. These new scales can be used for measuring, instead of meters or seconds. In this way we can replace the original r dimensions by r new dimensions from (combinations of) r variables. These r variables, when measured in the new dimensions, are by definition equal to unity, and play no visible role anymore. The remaining $n - r$ variables, on the other hand, may be expressed in the new dimensions to constitute the essential (and nondimensional) problem parameters.

Example. A problem with the 4 variables force F, length L, velocity V and viscosity η are expressed in 3 dimensions kg, m and s by $[F] = \mathrm{kg\,m/s^2}$, $[L] = \mathrm{m}$, $[V] = \mathrm{m/s}$ and $[\eta] = \mathrm{kg/m\,s}$.

With the inherent unit of length L, inherent unit of time L/V and inherent unit of mass $\eta L^2/V$, the variables L, V and η become simply 1 (times L, V and η, respectively). Only force F becomes some (dimensionless) number $\mathcal{F}$ times the new units as follows:

$$F = \mathcal{F} \cdot \frac{\dfrac{\eta L^2}{V} \cdot L}{\left(\dfrac{L}{V}\right)^2} = \mathcal{F} \cdot LV\eta, \qquad \text{in other words} \qquad \mathcal{F} = \frac{F}{LV\eta}.$$

A more formal way to obtain this is by utilizing a bit linear algebra. We have for any dimensionless quantity G the condition that it should satisfy for some combination of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

$$[G] = [F^{\alpha_1} L^{\alpha_2} V^{\alpha_3} \eta^{\alpha_4}] = \mathrm{m}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4}\,\mathrm{kg}^{\alpha_1+\alpha_4}\,\mathrm{s}^{-2\alpha_1-\alpha_3-\alpha_4} = \mathrm{m}^0\,\mathrm{kg}^0\,\mathrm{s}^0 = 1.$$

In other words we have $r = 3$ equations for $n = 4$ unknowns

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ -2 & 0 & -1 & -1 \end{pmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

[1] In [1] it is incorrectly stated *at most*.

Since all equations are independent, this system has rank $= 3$, the number of equations r, and so $4 - 3 = 1$ linearly independent solutions. Therefore, there is one dimensionless variable G. (If some rows are dependent, the rank would have been less than r and the number of independent solutions more than $n - r$.) Solving this system yields the solution $\boldsymbol{\alpha} = (1, -1, -1, -1)$, or any multiple of it. The corresponding dimensionless number is then

$$G = \frac{F}{LV\eta},$$

which confirms the above result with $G = \mathcal{F}$. Note that other forms, like G^2, $\sqrt{G}$, $1/G$ etc. are equally possible dimensionless numbers, equivalent to G.

2.1.4.2 Weber's Law.

Normally, we have in the problems studied several variables and parameters of the same unit (dimension), which act as each other's reference to compare with. The opposite situation, when there is *no* reference available, is also meaningful.

When a variable is perceived for which there is no reference quantity available to compare with, c.q. to scale on, the actual value of the variable itself will be the reference. The resulting logarithmic relation (see below) is known as *Weber's Law*[2].

Take for example the perceived loudness of sound. Since the range of our human audible sensitivity is incredibly large (10^{14} in energy), the loudest and quietest levels are practically infinitely far away. Therefore, we have no reference or scaling level to compare with, other than the actually perceived sound itself.

As a result, variations in sound loudness $\mathrm{d}L$ are perceived proportional to *relative* variations of the physical sound intensity $\mathrm{d}I/I$:

$$\mathrm{d}L = K\frac{\mathrm{d}I}{I},$$

for a suitably chosen constant K. After integration we obtain that L varies logarithmically in I.

$$L = L_0 + K \log I$$

with L_0 a conveniently chosen reference level.

As the intensity (the time-averaged energy flux) I is, for a single tone, proportional to the mean squared acoustic pressure $p_{\rm rms}^2$, we have the relation $L = K \log(p_{\rm rms}^2) + L_0$. If

$$L = 2\log_{10}(p_{\rm rms}/p_0)$$

for a reference value $p_0 = 2\cdot 10^{-5}$ Pascal is taken, we call L the Sound Pressure Level in Bells. The usual unit is one tenth of it, the decibel.

[2] Ernst Heinrich Weber, 1834

2.1.5 Example: a systematic derivation of the Korteweg-de Vries Equation

Introduction

The *Korteweg-de Vries* equation describes weakly nonlinear, weakly dispersive long water waves, *i.e.* surface waves with gravity as the restoring force on a inviscid, incompressible, irrotational steady mean flow with negligible surface tension and a constant horizontal bottom.

The derivation of the equation is not trivial and the number of assumptions is quite large. In most derivations given in the literature these assumptions are not all or not explicitly given.

The problem

Consider the two-dimensional space $-\infty < x < \infty$ and $0 < y < h + \eta(x,t)$ filled with water with velocity $\boldsymbol{v} = \nabla\phi$, constant density ρ_0, pressure p and water surface $y = h + \eta$. The dynamics of the water is given by the equations

$$\phi_{xx} + \phi_{yy} = 0 \tag{2.1}$$

$$\rho_0\phi_t + \tfrac{1}{2}\rho_0\big(\phi_x^2 + \phi_y^2\big) + p + \rho_0 g y = C(t) \tag{2.2}$$

and boundary conditions

$$\phi_y = 0 \qquad \text{at } y = 0 \tag{2.3}$$

$$p = p_0 \qquad \text{at } y = h + \eta \tag{2.4}$$

$$\phi_y = \eta_x\phi_x + \eta_t \qquad \text{at } y = h + \eta \tag{2.5}$$

where p_0 is the assumed constant atmospheric pressure above the water surface.

Equation (2.1) results from mass conservation; equation (2.2) is Bernoulli's equation or integrated momentum equation with arbitrary integration constant $C(t)$, which will be chosen here equal to $p_0 + \rho_0 g h$; condition (2.3) describes the hard walled bottom; condition (2.4) describes the continuity of pressure across the water surface; condition (2.5) describes the fact that the water surface is a streamline. This last equation can be derived as follows. Assume a water particle with position $(x, y) = (X(t), Y(t))$ following the surface streamline, and thus satisfying

$$Y(t) = h + \eta(X(t), t). \tag{2.6}$$

Then condition (2.5) follows by differentiation and noting that

$$\boldsymbol{v}(X(t), Y(t)) = \left(\frac{\mathrm{d}X}{\mathrm{d}t}, \frac{\mathrm{d}Y}{\mathrm{d}t}\right). \tag{2.7}$$

Anticipating a smooth solution and relatively small perturbations, we expand the conditions at $y = h + \eta$ around $\eta = 0$ and convert them into conditions at $y = h$:

$$\phi_y + \eta\phi_{yy} + \tfrac{1}{2}\eta^2\phi_{yyy} + \cdots = \eta_x\phi_x + \eta\eta_x\phi_{xy} + \cdots + \eta_t \tag{2.8}$$

$$\begin{aligned}\phi_t + \eta\phi_{ty} + \tfrac{1}{2}\eta^2\phi_{tyy} + \cdots + \tfrac{1}{2}\phi_x^2 + \eta\phi_x\phi_{xy} + \tfrac{1}{2}\eta^2\big(\phi_{xy}^2 + \phi_x\phi_{xyy}\big) + \ldots \\ + \tfrac{1}{2}\phi_y^2 + \eta\phi_y\phi_{yy} + \tfrac{1}{2}\eta^2\big(\phi_{yy}^2 + \phi_y\phi_{yyy}\big) + \cdots + g\eta = 0.\end{aligned} \tag{2.9}$$

Scaling and non-dimensionalisation

Assume that the typical length scale in x-direction of the waves to be considered is L, and the typical amplitude of the perturbed surface is a. Assume that a is small compared to water depth h, and h is small compared to L, in such a way that $aL^2/h^3 = O(1)$. In other words, if we introduce the small parameters

$$\varepsilon = \frac{a}{h}, \qquad \delta = \left(\frac{h}{L}\right)^2 \tag{2.10}$$

it is assumed that $\varepsilon/\delta = O(1)$.

We further assume that variation of ϕ in y scale on h. By trial and error it appears that typical variations in time scale on

$$T = \frac{L}{\sqrt{gh}} \tag{2.11}$$

for the waves considered. With the above considerations we scale our variables to dimensionless form

$$\phi := agT\phi, \quad \eta := a\eta, \quad x := Lx, \quad y := hy, \quad t := Tt \tag{2.12}$$

as follows. First we have the boundary conditions at $y = 1$.

$$\phi_y + \varepsilon\eta\phi_{yy} + \tfrac{1}{2}\varepsilon^2\eta^2\phi_{yyy} + \cdots = \varepsilon\delta\eta_x\phi_x + \cdots + \delta\eta_t \tag{2.13}$$

$$\begin{aligned}\phi_t + \varepsilon\eta\phi_{ty} + \tfrac{1}{2}\varepsilon^2\eta^2\phi_{tyy} + \cdots + \tfrac{1}{2}\varepsilon\phi_x^2 + \varepsilon^2\eta\phi_x\phi_{xy} + \tfrac{1}{2}\varepsilon^3\eta^2\big(\phi_{xy}^2 + \phi_x\phi_{xyy}\big) + \dots \\ +\tfrac{1}{2}\varepsilon\delta^{-1}\phi_y^2 + \varepsilon^2\delta^{-1}\eta\phi_y\phi_{yy} + \tfrac{1}{2}\varepsilon^3\delta^{-1}\eta^2\big(\phi_{yy}^2 + \phi_y\phi_{yyy}\big) + \cdots + \eta = 0.\end{aligned} \tag{2.14}$$

Then we have the equation in $-\infty < x < \infty$, $0 < y < 1$

$$\delta\phi_{xx} + \phi_{yy} = 0 \tag{2.15}$$

with boundary condition at $y = 0$

$$\phi_y = 0. \tag{2.16}$$

Asymptotic analysis

If we substitute the expansion

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \dots \tag{2.17}$$

we get

$$\phi_{0,yy} + \delta(\phi_{0,xx} + \phi_{1,yy}) + \delta^2(\phi_{1,xx} + \phi_{2,yy}) + \cdots = 0 \tag{2.18}$$

With the hard-wall boundary condition this results into

$$\phi_0 = \psi(x,t) \tag{2.19}$$

$$\phi_1 = A_1(x,t) - \tfrac{1}{2}y^2\psi_{xx}(x,t) \tag{2.20}$$

$$\phi_2 = A_2(x,t) - \tfrac{1}{2}y^2A_{1,xx} + \tfrac{1}{24}y^4\psi_{xxxx}(x,t) \tag{2.21}$$

Substitute these results together with the expansion

$$\eta = \eta_0 + \delta\eta_1 + \delta^2\eta_2 + \dots \tag{2.22}$$

into (2.13) and (2.14), then we obtain to leading orders

$$-\psi_{xx} - \delta A_{1,xx} + \tfrac{1}{6}\delta\psi_{xxxx} - \varepsilon\eta_0\psi_{xx} = \varepsilon\eta_{0,x}\psi_x + \eta_{0,t} + \delta\eta_{1,t} \tag{2.23}$$

$$\psi_t + \delta A_{1,t} - \tfrac{1}{2}\delta\psi_{xxt} + \tfrac{1}{2}\varepsilon\psi_x^2 + \eta_0 + \delta\eta_1 = 0. \tag{2.24}$$

For notational convenience we introduce

$$w(x,t) = \psi(x,t) + \delta A_1(x,t), \quad \zeta(x,t) = \eta_0(x,t) + \delta\eta_1(x,t). \tag{2.25}$$

Then we have to the same order of accuracy

$$\zeta_t + w_{xx} = \tfrac{1}{6}\delta w_{xxxx} - \varepsilon\zeta w_{xx} - \varepsilon\zeta_x w_x, \tag{2.26}$$

$$\zeta + w_t = \tfrac{1}{2}\delta w_{xxt} - \tfrac{1}{2}\varepsilon w_x^2. \tag{2.27}$$

Further assumptions

It is easily verified that to leading order both ζ and w satisfy the wave equation

$$\zeta_{tt} - \zeta_{xx} = 0, \quad w_{tt} - w_{xx} = 0 \tag{2.28}$$

with solutions any linear combination of right running wave $F(x-t)$ and a left running wave $G(x+t)$. For the nonlinear problem this is not productive because we look for slow modulations on a right or left running wave, whereas a combination would produce kinematically non-essential fast variations. So we limit ourselves to solutions of the form

$$\zeta := \zeta(z,\tau), \quad w := w(z,\tau), \quad \text{where} \quad z := x - t,\ \tau := \delta t, \tag{2.29}$$

We obtain to the same order of accuracy

$$-\zeta_z + \delta\zeta_\tau + w_{zz} = \tfrac{1}{6}\delta w_{zzzz} - \varepsilon\zeta w_{zz} - \varepsilon\zeta_z w_z, \tag{2.30}$$

$$\zeta - w_z + \delta w_\tau = -\tfrac{1}{2}\delta w_{zzz} - \tfrac{1}{2}\varepsilon w_z^2. \tag{2.31}$$

Getting the equation

From (2.31) we have

$$w_z = \zeta + \delta w_\tau + \tfrac{1}{2}\delta w_{zzz} + \tfrac{1}{2}\varepsilon w_z^2 \tag{2.32}$$

If we substitute this expression for w_z into (2.30)

$$-\zeta_z + \delta\zeta_\tau + \zeta_z + \delta w_{z\tau} + \tfrac{1}{2}\delta w_{zzzz} + \varepsilon w_z w_{zz} = \tfrac{1}{6}\delta w_{zzzz} - \varepsilon\zeta w_{zz} - \varepsilon\zeta_z w_z \tag{2.33}$$

or

$$\zeta_\tau + w_{z\tau} + \tfrac{1}{3}w_{zzzz} + \varepsilon\delta^{-1}\big(w_z w_{zz} + \zeta w_{zz} + \zeta_z w_z\big) = 0 \tag{2.34}$$

and again use $w_z = \zeta + \ldots$, we can eliminate w completely from the equation and obtain to the same order of accuracy

$$\zeta_\tau + \tfrac{1}{6}\zeta_{zzz} + \tfrac{3}{2}\varepsilon\delta^{-1}\zeta\zeta_z = 0 \tag{2.35}$$

which is (a version of) the celebrated Korteweg-de Vries equation. If we like clean equations, we can transform in various ways

$$\zeta(z,\tau) = \lambda\sigma(\alpha z, \beta\tau) \tag{2.36}$$

(for example $\lambda = \frac{1}{9}\delta\varepsilon^{-1}$, $\alpha = 1$, $\beta = \frac{1}{6}$) to get

$$\sigma_2 + \sigma_{111} + \sigma\sigma_1 = 0 \tag{2.37}$$

2.2 Modelling, Nondimensionalisation and Scaling: Assignments

2.2.1 Travel time in cities

A simple model for the travel time by car between two addresses in a big city is: the time T in minutes is equal to the distance L in kilometers plus the number N of traffic lights passed,

$$T = L + N.$$

a) What is this formula if time is measured in hours and distance in miles?
b) Generalise the formula for arbitrary units of time and length.
c) Make this last version dimensionless in a suitable way.

2.2.2 Membrane resonance.

The resonance frequency ω of a freely suspended membrane (like a framedrum, a skin stretched over a frame without a resonance cavity) is determined by the membrane tension T, membrane surface density σ, membrane diameter a, air density ρ_a and sound speed c_a. In other words, there is a relation

$$\omega = f(T, \sigma, a, \rho_a, c_a).$$

According to Buckingham, this relation can be reduced to a relation between three dimensionless groups:

$$\left.\begin{array}{llll} \text{frequency} & \omega\ , & \text{dimension} & 1/\text{s} \\ \text{memb. tension} & T\ , & \text{"} & \text{kg/s}^2 \\ \text{memb. density} & \sigma, & \text{"} & \text{kg/m}^2 \\ \text{memb. diameter} & a\ , & \text{"} & \text{m} \\ \text{air density} & \rho_a, & \text{"} & \text{kg/m}^3 \\ \text{air soundspeed} & c_a\ , & \text{"} & \text{m/s} \end{array}\right\} \text{Buckingham: } 6-3=3 \text{ dimensionless groups } G$$

$$G = \omega^{\alpha_1} T^{\alpha_2} \sigma^{\alpha_3} a^{\alpha_4} \rho_a^{\alpha_5} c_a^{\alpha_6}$$

$$\begin{aligned}[G] &= \left(\frac{1}{\text{s}}\right)^{\alpha_1} \left(\frac{\text{kg}}{\text{s}^2}\right)^{\alpha_2} \left(\frac{\text{kg}}{\text{m}^2}\right)^{\alpha_3} \text{m}^{\alpha_4} \left(\frac{\text{kg}}{\text{m}^3}\right)^{\alpha_5} \left(\frac{\text{m}}{\text{s}}\right)^{\alpha_6} \\ &= \text{m}^{-2\alpha_3+\alpha_4-3\alpha_5+\alpha_6} \text{s}^{-\alpha_1-2\alpha_2-\alpha_6} \text{kg}^{\alpha_2+\alpha_3+\alpha_5} = \text{m}^0\text{s}^0\text{kg}^0\end{aligned}$$

a) Give (mutually independent) examples of the 3 possible dimensionless numbers G.
b) Show that it is possible to write the functional dependence between the frequency and the other parameters as

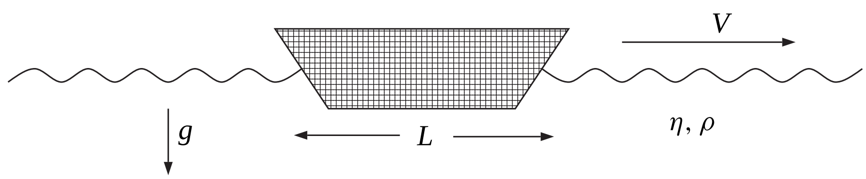
$$G_\omega = F(G_1, G_2)$$

where G_ω is the only parameter that depends on ω. You may introduce for convenience $c_M = (T/\sigma)^{\frac{1}{2}}$, the propagation speed of transversal waves in the membrane in the absence of air loading.

2.2.3 Ship drag: wave and viscosity effects.

A ship of typical size L, moving with velocity V in deep water of density ρ and viscosity η, feels a drag D due to gravity waves and due to viscous friction, apart from density, velocity and geometry effects. Symbolically, we have

$$D = f(g, \eta, \rho, V, L).$$



According to Buckingham, this relation can be reduced to a relation between three dimensionless groups:

$$\left.\begin{array}{llll} \text{drag} & D\,, & \text{dimension} & \mathrm{kg\,m/s^2} \\ \text{length} & L\,, & \text{''} & \mathrm{m} \\ \text{velocity} & V\,, & \text{''} & \mathrm{m/s} \\ \text{viscosity} & \eta, & \text{''} & \mathrm{kg/m\,s} \\ \text{gravity} & g\,, & \text{''} & \mathrm{m/s^2} \\ \text{water density} & \rho, & \text{''} & \mathrm{kg/m^3} \end{array}\right\} \quad \text{Buckingham: } 6-3=3 \text{ dimensionless groups } G$$

$$G = D^{\alpha_1} L^{\alpha_2} V^{\alpha_3} \eta^{\alpha_4} g^{\alpha_5} \rho^{\alpha_6}$$

$$\begin{aligned}[G] &= \left(\frac{\mathrm{kg\,m}}{\mathrm{s}^2}\right)^{\alpha_1} \mathrm{m}^{\alpha_2} \left(\frac{\mathrm{m}}{\mathrm{s}}\right)^{\alpha_3} \left(\frac{\mathrm{kg}}{\mathrm{m\,s}}\right)^{\alpha_4} \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)^{\alpha_5} \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)^{\alpha_6} \\ &= \mathrm{m}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4+\alpha_5-3\alpha_6} \mathrm{s}^{-2\alpha_1-\alpha_3-\alpha_4-2\alpha_5} \mathrm{kg}^{\alpha_1+\alpha_4+\alpha_6} = \mathrm{m}^0\mathrm{s}^0\mathrm{kg}^0\end{aligned}$$

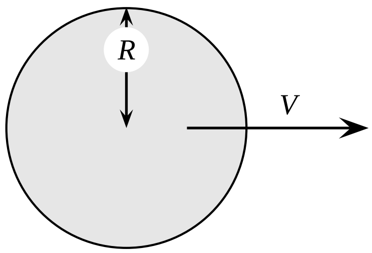
a) Give (mutually independent) examples of the 3 possible dimensionless numbers G.
b) Show that it is possible to write the functional dependence between the drag and the other parameters as

$$G_D = F(G_g, G_\eta)$$

where G_D is a parameter that depends on D but not on g or η, G_g depends on g but not on D or η, and G_η depends on η but not on D or g.

2.2.4 Sphere in viscous flow

Work out in detail – using Buckingham's theorem – scaling and non-dimensionalisation of the problem of the viscous air resistance (drag D, velocity V) of a sphere (radius R) in a fluid (density ρ, viscosity η). What would be a suitable scaling if viscosity dominates the resistance? And what if pressure difference dominates?



Sphere in viscous fluid

$$\left.\begin{array}{llcl} \text{length} & R, & \text{dimension} & \text{m} \\ \text{velocity} & V, & \text{''} & \text{m/s} \\ \text{viscosity} & \eta, & \text{''} & \text{kg/m s} \\ \text{density} & \rho, & \text{''} & \text{kg/m}^3 \end{array}\right\} \quad D = f(\rho, V, \eta, R)$$

2.2.5 Cooling of a cup of tea.

The total amount of thermal energy in a cup of tea of volume V, water density ρ, specific heat c and temperature T at time t is $E(t) = \rho c V T(t)$. According to Newton's cooling law, the heat flux through the surface A is $q = -hA(T - T_\infty)$ with heat transfer coefficient h. What is the dimension of h? Make the problem dimensionless and determine the characteristic time scale of the problem.

Confirm this by solving the equation for the decaying temperature $T(t)$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = q, \qquad T(0) = T_0.$$

2.2.6 The velocity of a rowing boat.

Determine the functional dependence of the velocity v of a rowing boat on the number n of rowers by using the following modelling assumptions.

The size of the boat scales with the number of rowers (*i.e.* their volume) but has otherwise the same shape. So if the volume per rower is G, the volume of the boat is $V = nG$. Furthermore, the volume of the boat can be written as a length L times a cross section $A = \ell^2$ and $L = \lambda\ell$ for a shapefactor λ.

The drag only depends on the water pressure distribution and is for high enough Reynolds numbers given by $D = \frac{1}{2}\rho v^2 A C_D$, where ρ is the water density and C_D the drag coefficient, which is a constant as it depends only on the shape of the boat.

The required thrust is therefore $F = D$, while the necessary power to maintain the velocity v is then $P = \frac{\mathrm{d}}{\mathrm{d}t}\int^x F\,\mathrm{d}x' = Fv$. The available power per rower is a fixed p.

2.2.7 A sessile drop with surface tension.

The height h of a drop of liquid at rest on a horizontal surface with the effect of gravity being balanced by surface tension is a function of liquid density ρ, volume L^3, acceleration of gravity g, surface tension γ and contact angle θ. As $[h] = $ m, $[\rho] = $ kg/m^3, $[L] = $ m, $[g] = $ m/s^2, $[\gamma] = $ kg/s^2, and $[\theta] = 1$, we have $6 - 3 = 3$ dimensionless numbers. One is of course the already dimensionless θ. The second dimensionless number is the Bond number, known to control this kind of problems, and is given by

$$B = \frac{\rho g L^2}{\gamma}.$$

The third is a dimensionless number containing h, leading to a functional relationship given by

$$h = \ell F(B, \theta),$$

where F is dimensionless and ℓ is an inherent length scale. We have practically two useful choices for ℓ. One is suitable when B is small (high relative surface tension) and the drop becomes spherical. The other is the proper scaling when B is large (low relative surface tension), such that the drop will spread out, flat as a pancake, and $h \ll L$. In particular, $h/L = O(B^{-1/2})$

Find these two (mutually independent) possible ℓ_1 and ℓ_2.

2.2.8 The drag of a plate sliding along a thin layer of lubricant.

Find a functional relation for the drag D of a plate of size $L \times W$ slipping with velocity V along a thin layer of grease of thickness h and viscosity η. Assume that the drag is linearly proportional to the wetted surface.

length	L,	dimension	m
width	W,	”	m
velocity	V,	”	m/s
viscosity	η,	”	kg/m s
thickness	h,	”	m

2.2.9 The suspended cable

A cable, suspended between the points $X = 0$, $Y = 0$ and $X = D$, $Y = 0$, is described as a linear elastic, geometrically non-linear inextensible bar[3] of bending stiffness EI and weight Q per unit length.

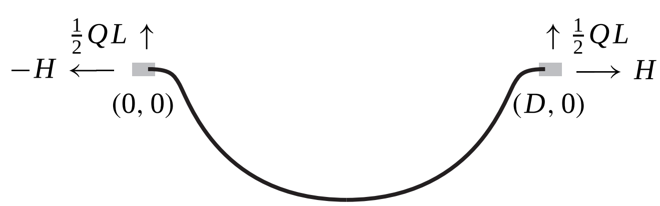


Figure 2.1: A suspended cable

At the suspension points the cable is horizontally clamped such that the cable hangs in the vertical plane through the suspension points. The total length L of the cable is larger than D, so the cable is not stretched.

In order to keep the cable in position, the suspension points apply a reaction force, with horizontal component $-H$ resp. H, and a vertical component V, resp. $QL - V$. From symmetry we already have $V = QL - V$ so $V = \frac{1}{2}QL$ is known. On the other hand, H, the force that keeps the cable ends apart, is unknown.

[3] A so called Euler-Bernoulli bar.

Let s be the arc length along the cable, and $\psi(s)$ the tangent angle with the horizon. Then the cartesian co-ordinates $(X(s), Y(s))$ of a point on the cable are given by

$$X(s) = \int_0^s \cos\psi(s')\,\mathrm{d}s', \quad Y(s) = \int_0^s \sin\psi(s')\,\mathrm{d}s'.$$

The shape of the cable $\psi(s)$ and the necessary force H, are determined by the following differential equation and boundary conditions

$$EI\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} = H\sin\psi - (Qs - V)\cos\psi$$

$$\psi(0) = 0, \quad \psi(L) = 0, \quad X(L) = D, \quad Y(L) = 0.$$

a. Make the equations and boundary conditions dimensionless by scaling all lengths on L.
How many (and which) dimensionless problem parameters do we have? How does this conform to Buckingham's Theorem?
b. Under what conditions can we approximate the equation by

$$0 = H\sin\psi - (Qs - V)\cos\psi.$$

Can we keep all the boundary conditions? Which would you keep? Solve the remaining equation.
c. Under what conditions can we approximate the equation by

$$EI\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} = H\psi - (Qs - V).$$

Can we keep all the boundary conditions? Do we have to adapt any to bring it in line with the used approximation? Can you solve the remaining equation (up to a numerical evaluation)?

2.2.10 Electrically heated metal

A piece of metal Ω of size L is heated, from an initial state $T(\boldsymbol{x}, t) \equiv 0$, to a temperature distribution T by applying at $t = 0$ an electric field with potential ψ and typical voltage V (Fig. 2.2). This heat

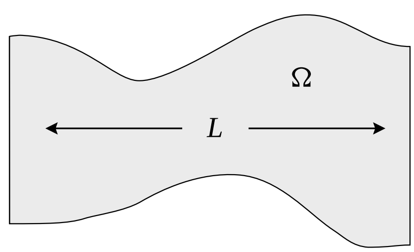


Figure 2.2: A piece of metal heated by an electric field.

source, the energy dissipation of the electric field, is given by the inhomogeneous term $\sigma|\nabla\psi|^2$ in the following inhomogeneous heat equation

$$C\frac{\partial T}{\partial t} = \kappa\nabla^2 T + \sigma|\nabla\psi|^2.$$

The edges are kept at $T = 0$, yielding a dissipation of thermal energy. As time proceeds, the temperature distribution will converge to a steady state corresponding to an equilibrium of heat production by the source and heat loss via the edges. We are interested in the typical time this takes and the typical final temperature.

If we introduce the formal scaling $T = T_0 u$, $t = t_0 \tau$, $\boldsymbol{x} = L\boldsymbol{\xi}$, and $\psi = V\Psi$, then we get

$$\frac{CT_0}{t_0}\frac{\partial u}{\partial \tau} = \frac{\kappa T_0}{L^2}\nabla_{\xi}^2 u + \frac{\sigma V^2}{L^2}|\nabla_{\xi}\Psi|^2.$$

a. If we take the final (steady state) situation as reference, what would then be our choice for T_0?
b. What is then the choice for the time t_0?

Note that the boundary conditions are rather important. If the edges were thermally isolated, we would, at least initially, have no temperature gradients scaling on L, and the diffusion term $\kappa\nabla^2 T$ would be negligible. Only the storage term $C\frac{\partial}{\partial t}T$ would balance the source term, and there would be no other temperature to scale on than $\sigma V^2 t_0/CL^2$. In other words, the temperature would rise approximately linearly in time.

2.2.11 Traffic waves

A simple (but nonlinear) one-dimensional wave equation, used (for example) to model traffic flow density ρ at position x and time t, is

$$\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = 0, \qquad \rho(x,0) = F(x).$$

Since dimensional quantities must include an inherent scale, we can write (with dimensionless shape functions g and f)

$$C(u) = C_0 g\left(\frac{\rho}{D}\right), \qquad F(x) = \rho_0 f\left(\frac{x}{L}\right).$$

a. Make the problem dimensionless in a sensible way. What is the remaining dimensionless parameter?
b. Show that the solution ρ is implicitly given by

$$\rho = F(x - C(\rho)t).$$

It is sufficient to consider the original equation. The dimensionless solution is similar.

2.2.12 The Korteweg-de Vries equation

A version of the Korteweg-de Vries equation (an equation for certain types of water waves) is given by

$$A\zeta_t + B\zeta_{xxx} + C\zeta\zeta_x = 0$$

Rescale the $\zeta = \lambda\sigma$, $x = \alpha z$ and $t = \beta\tau$, such that the remaining equation has only coefficients equal to 1.

2.2.13 Just an equation

x satisfies the following equation

$$ax^2 + bf\left(\frac{x}{L}\right) = 0$$

with parameters a, b and L, and dimensionless function f with dimensionless argument, while $[x] =$ meters and $[b] =$ seconds.

a) What are the dimensions of a and L?
b) Find, by scaling $x = \lambda X$ for some suitable λ and collecting parameters in dimensionless groups R, equivalent equations of the form

$$X^2 + Rf(X) = 0, \quad X^2 + f(RX) = 0.$$

c) Under what conditions can the original equation be approximated by

$$f\left(\frac{x}{L}\right) = 0$$

2.2.14 The pendulum

Consider a pendulum consisting of a bob of mass m, suspended from a fixed, massless support of length L. The acceleration of gravity is g. Depending on time variable t, the pendulum angular displacement $\phi(t)$ swings between angle $-\alpha$ and α.

angle	ϕ,	dimension	-
angle	α,	”	-
time	t,	”	s
mass	m,	”	kg
length	L,	”	m
gravity	g,	”	m/s^2

a) What is the inherent time scale of the problem?
b) The motion is given by the equation

$$mL\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + mg\sin\phi = 0.$$

Using a), make this equation dimensionless.
c) Under what condition can we approximate the dimensionless equation by

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} + \phi - \tfrac{1}{6}\phi^3 = 0$$

2.2.15 Heat convection and diffusion

Consider a steady flow field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ of air of uniform density ρ and specific heat capacity c, and temperature $T = T(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ and time t. The heat is convected by the flow and diffused by Fourier's law for heat conduction, leading to the equations

$$\rho c \left(\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T \right) = -\nabla \cdot \boldsymbol{q}, \quad \boldsymbol{q} = -\kappa \nabla T,$$

where $\boldsymbol{q}$ is the heat flux density and κ is the coefficient of conductivity.

Assume that the typical velocity of the velocity field is U_0, and the length scale of the variation of both the flow field and the temperature field is L. Neglecting transient effects we have thus a typical time scale of L/U_0.

temperature	T,	dimension	K
length scale	L,	”	m
velocity	U_0,	”	m/s
density	ρ,	”	kg/m^3
heat flux density	$\boldsymbol{q}$,	”	W/ m^2
specific heat capacity	c ,	”	J/kgK
conductivity	κ,	”	W/mK

a) Under what conditions (*i.e.* for which small parameter) can the diffusion be neglected, such that we obtain the simplified equation

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = 0$$

b) Show that (under these conditions) the temperature is constant along any streamline $\boldsymbol{x} = \boldsymbol{\xi}(t)$, given by

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}.$$

2.2.16 Heat conduction in a long bar

A semi-infinite isolated metal bar, given by $0 \leqslant x < \infty$, is heated by a uniform heat source of constant flux density Q at $x = 0$, starting from $t = 0$. Assume that the initial temperature $T = 0$, such that T is linearly proportional to Q. The bar metal has a specific heat capacity c, density ρ and conductivity κ. Due to the uniform source and the isolation, the temperature along a cross section is uniform.

temperature	T,	dimension	K
length	x,	”	m
time	t,	”	s
density	ρ,	”	kg/m^3
specific heat capacity	c ,	”	J/kgK
conductivity	κ,	”	W/mK
heat source	Q,	”	W/m^2

a) According to Buckingham's Π theorem, there are $6 - 4 = 2$ dimensionless groups possible (note that $T \propto Q$, so T/Q is to be considered as one variable). Give examples of such groups.

b) Show, by using a), that the most general form for $T(x,t)$ is[4]

$$T(x,t) = \frac{Qx}{\kappa} F\left(\sqrt{\frac{x^2 \rho c}{\kappa t}}\right).$$

c) Assume that T satisfies the equation

$$\rho c \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

and define the *similarity variable* $\eta = \sqrt{x^2 \rho c / \kappa t}$. Derive the (ordinary) differential equation in the variable η for function $F(\eta)$ of b). Use the chain rule carefully when differentiating T to x and t. Make sure that the final equation *only* depends on η and contains no x or t dependence anymore.

The solution of this equation is not standard but can be found (for example) by Mathematica or Wolfram Alpha.

2.2.17 A simple balloon

A balloon rises in the atmosphere of density ρ_a such that it is at height $h(t)$ at time t. The balloon of mass m, fixed volume V and cross sectional surface A is subject to inertia $-mh''$, Archimedean (buoyancy) force $g\rho_a V$, weight $-mg$ and air drag $-\frac{1}{2}\rho_a C_d A(h')^2$, where $g = 9.8$ m/s^2 is the acceleration of gravity, and drag coefficient C_d depends on the geometry but is for a sphere (and high enough Reynolds number) in the order of 0.5.

Together these forces cancel out each other, so altogether we have the following equation for the dynamics of the balloon

$$m\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = g\rho_a V - gm - \frac{1}{2}\rho_a \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 C_d A.$$

Assume that $h(0) = 0$ and $h'(0) = 0$. The atmospheric air density will vary (in the troposphere, *i.e.* for $0 \leqslant h \leqslant 11$ km) with the height according to

$$\rho_a(h) = \rho_0 \left(1 - \frac{h}{L}\right)^{\alpha} \text{ kg/m}^3, \quad \text{with } \rho_0 = 1.225\,\text{kg/m}^3, \quad L = 44.33\,\text{km}, \quad \alpha = 4.256.$$

In practice a flexible balloon will grow in size with the decreasing atmospheric pressure, but we will ignore this and assume that the material is very stiff.

Make the equation dimensionless on the inherent length and time scales. There are two natural length scales in the problem (the atmospheric variation L and the diameter of the balloon $\sim V^{3/2}$, $\sim A^{1/2}$). What seems to be the most sensible one? Try both if you hesitate. The suitable time scale can be found by assuming that the dynamics is dominated by the balance between the buoyancy and the drag. When is this possibly not the case?

[4] The seemingly different $T(x,t) = (Qt/\rho c x) G\big(\sqrt{x^2 \rho c / \kappa t}\big)$ is in reality of the same form. Write $F(\eta) = \eta^{-2} G(\eta)$.

Introduce convenient dimensionless parameters and (in the case of ρ_a) shape function. Can you interpret these parameters? For what conditions can we neglect the inertia term? Is this reasonable for a balloon of $m = 1$ kg, $V = 2$ m^3 and $A = 1.9$ m^2. What about the initial conditions? The remaining equation is still difficult, but can you solve it if you assume that $m/\rho_0 V$ is small, while $1 - h/L$ is not small?

2.2.18 A pulsating sphere

The radially symmetric sound field of a pulsating sphere $r = a_0 + a(t)$ (with a small) in a medium with mean density ρ_0 and sound speed c_0 is described by the following (linearised) equations for pressure perturbation p, density perturbation ρ and velocity perturbation v.

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \rho_0 \left(\frac{\partial v}{\partial r} + 2\frac{v}{r} \right) &= 0, \\ \rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial r} &= 0, \\ p - c_0^2 \rho &= 0.\end{aligned}$$

while

$$v = \frac{\partial a}{\partial t} \quad \text{at } r = a_0.$$

If the sphere pulsates harmonically with frequency ω, we write for convenience

$$a = \mathrm{Re}(\hat{a}\, \mathrm{e}^{\mathrm{i}\omega t}),\ p = \mathrm{Re}(\hat{p}\, \mathrm{e}^{\mathrm{i}\omega t}),\ v = \mathrm{Re}(\hat{v}\, \mathrm{e}^{\mathrm{i}\omega t}),\ \rho = \mathrm{Re}(\hat{\rho}\, \mathrm{e}^{\mathrm{i}\omega t}).$$

leading to the equations (we eliminate ρ)

$$\begin{aligned}\mathrm{i}\omega \hat{p} + \rho_0 c_0^2 \left(\frac{\partial \hat{v}}{\partial r} + 2\frac{\hat{v}}{r} \right) &= 0, \\ \mathrm{i}\omega \rho_0 \hat{v} + \frac{\partial \hat{p}}{\partial r} &= 0.\end{aligned}$$

with

$$\hat{v} = \mathrm{i}\omega \hat{a} \quad \text{at} \quad r = a_0.$$

The proper solution of the equations can be shown to be

$$\begin{aligned}\hat{p} &= \frac{A}{r}\, \mathrm{e}^{-\mathrm{i}kr} \\ \hat{v} &= \frac{1}{\rho_0 c_0} \frac{A}{r} \left(1 + \frac{1}{\mathrm{i}kr} \right) \mathrm{e}^{-\mathrm{i}kr}\end{aligned}$$

with constant A to be determined, and the acoustic wavenumber

$$k = \frac{\omega}{c_0} = \frac{2\pi}{\lambda}$$

where λ is the free field wavelength.

a. Determine A by applying the boundary condition at $r = a_0$.
b. Scale $\hat{p}$ and $\hat{v}$ on $\hat{a}/a_0$, and make dimensionless: $\hat{p}$ on ρc_0^2 and $\hat{v}$ on c_0. Lengths can be scaled on a_0 and on $1/k$. Do both. Their ratio, dimensionless number $\varepsilon = ka_0$, is called Helmholtz number.
c. Simplify the formulas for small source size (known as a *compact source*), *i.e.* $\varepsilon = ka_0 \ll 1$. What do you get in each case of scaling? Can you interpret the results?

2.2.19 Similarity solutions for non-linear and linear diffusion

Consider the temperature T due to a heat source at $\boldsymbol{r} = \boldsymbol{0}$ in a spherically symmetric environment of specific heat capacity c_p, density ρ and conductivity κ.

All parameters are constant, except κ which is a function of the absolute temperature T. We assume here $\kappa = \kappa_0 T^{n-1}$. For example, for diamond, $\kappa_0 = 27\,530$ W/mK^n and $n - 1 = -1.26$.

We have the equation

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot \left(\kappa \nabla T\right)$$

a) Scale $T = T_0 u$ such that we obtain
$$\frac{\partial u}{\partial t} = \nabla^2 u^n$$

b) Consider spherically symmetric similarity solutions of the form
$$u = u(r,t) = t^{\alpha} F(z), \quad z = r t^{-\beta}, \quad r = |\boldsymbol{x}|.$$
What are the restrictions on α and β? What is the remaining equation for F?

c) Find a solution of the form $F(z) = Cz^m$ for particular choice of m and C.

We continue with the more usual model of linear diffusion, *i.e.* where $n = 1$.

d) Scale time t such that we obtain for $T(r,t) = u(r,t')$ (we skip the prime in the following)
$$\frac{\partial u}{\partial t} = \nabla^2 u$$

e) Consider again spherically symmetric similarity solutions of the form
$$u = u(r,t) = t^{\alpha} F(z), \quad z = r t^{-\beta}, \quad r = |\boldsymbol{x}|.$$
What are the restrictions on α and β? What is the remaining equation for F? Find the general solution by using Maple.

f) Assume that the heat source is a source of constant flux Q, which corresponds to a condition
$$\int_{r=\varepsilon} -\kappa \nabla u \cdot \boldsymbol{n}\, \mathrm{d}S = -4\pi \varepsilon^2 \kappa \frac{\partial u}{\partial r}\Big|_{r=\varepsilon} = Q$$
for any sphere $r = \varepsilon$, in particular for $\varepsilon \to 0$. For what value of α is this condition satisfied? (Use Maple to find the behaviour of the integrand for small r.)

g) What is, for this choice of α, the resulting solution if we add the boundary condition that $u \to$ constant for $r \to \infty$? (Use Maple.)

2.2.20 Falling through the center of the earth

Although it is unlikely that such a tunnel will ever be excavated in the near future, we assume a vacuum straight tunnel right through the center of the earth. It connects two opposite points on the earth's surface, separated by the earth's diameter $2R$. If the earth's mass density ρ is uniform, then according to Newton's law of gravitation any object in the tunnel at radial position r is attracted only by the part of the earth's mass that is inside the concentric sphere of radius r. The proportionality constant is the universal gravity constant G.

At time $t = 0$ at position $r = R$ we drop a stone of negligible mass (compared to the mass of the earth) with zero initial speed. We wait until the stone returns at time $t = T$ (about 84 minutes).

The problem parameters and variables, according to our model, are

radius	R,	dimension	m
position	r,	"	m
time	t,	"	s
return time	T,	"	s
density	ρ,	"	$\mathrm{kg/m^3}$
gravity constant	G,	"	$\mathrm{m^3/s^2kg}$

Show by dimensional arguments that T depends only on ρ and G, and not on R. In other words, at whatever depth we release the stone, the return time is the same.

2.2.21 Energy consumption of a car

Consider a car of mass m at position $x(t)$ and velocity $v(t) = x'(t)$ at time t, moving from $x = 0$ to $x = L$ in time $t = 0$ to $t = T$ along a road of height $h(x)$ at position x. The car is subject to acceleration force mv', gravity force $-mgh'(x)$, air drag $bv|v| = \frac{1}{2}\rho AC_D v|v|$ (where ρ is the density of air, A is the car's frontal area, and C_D is its drag coefficient), internal friction cv, and engine thrust $F(t)$. Assuming an always positive velocity, we have then the balance of forces

$$mv' + bv^2 + cv + mgh'(x) = F(t).$$

We study the extra energy consumption due to velocity fluctuations. by comparing the energy consumption for a steady velocity $v(t) = V_0 = L/T$ with a velocity fluctuating around average V_0.

The necessary energy is the work done from $x = 0$ to L, or the power Fv integrated from $t = 0$ to T.

$$E = \int_0^L F\,\mathrm{d}x = \int_0^T Fv\,\mathrm{d}t.$$

Check (by integrating the equation) that, if $v(0) = v(T)$ and $h(0) = h(L)$, the energy only depends on the friction terms, i.e. b and c, and therefore not on m.

Make time dimensionless as $t = T\tau$ and position as $x = Ls$. Since m plays no role, we make masses dimensionless on bL, and the other variables similarly. Assume $v(t) = V_0(1 + \varepsilon u(\tau))$ with ε small, $u(0) = u(1) = 0$, and u normalised by (without normalisation of u, ε is not defined)

$$\int_0^1 u^2\,\mathrm{d}\tau = 1.$$

Find the extra energy consumption due to the fluctuating velocity, in dimensionless form, to leading order in ε.

Chapter 3

Asymptotic Analysis

3.1 Asymptotic approximations and expansions

Before we can introduce the methods, we have to define our terminology of asymptotic approximations and asymptotic expansions. We will start with an intuitive description, followed by a pointwise enumeration.

3.1.1 Asymptotic approximations

In order to give a qualitative description of the behaviour of a function f with parameter ε near a point of interest, say $\varepsilon = 0$ (equivalent to any other value by a simple translation), we have the so-called order symbols O, o, and O_s; see Section 3.2. Often $\varepsilon = 0$ is the lower limit of a parameter range, and we have the tacit assumption that $\varepsilon \downarrow 0$.

Definition 3.1 *$\varphi(\varepsilon)$ is an asymptotic approximation to $f(\varepsilon)$ as $\varepsilon \to 0$ if*

$$f(\varepsilon) = \varphi(\varepsilon) + o(\varphi(\varepsilon)) \quad \textit{as} \;\; \varepsilon \to 0,$$

sometimes more compactly denoted by $f \sim \varphi$.

If f and φ depend on $\boldsymbol{x}$, this definition remains valid pointwise, *i.e.* for $\boldsymbol{x}$ fixed. It is, however, useful to extend the definition to uniformly valid approximations.

Definition 3.2 *Let $f(\boldsymbol{x};\varepsilon)$ and $\varphi(\boldsymbol{x};\varepsilon)$ be continuous functions for $\boldsymbol{x} \in \mathcal{D}$ and $0 < \varepsilon < a$. We call $\varphi(\boldsymbol{x};\varepsilon)$ a uniform asymptotic approximation to $f(\boldsymbol{x};\varepsilon)$ for $\boldsymbol{x} \in \mathcal{D}$ as $\varepsilon \to 0$, if for any positive number δ there is an ε_1 (independent of $\boldsymbol{x}$ and ε) such that*

$$|f(\boldsymbol{x};\varepsilon) - \varphi(\boldsymbol{x};\varepsilon)| \leqslant \delta|\varphi(\boldsymbol{x};\varepsilon)| \;\textit{for}\; \boldsymbol{x} \in \mathcal{D} \;\textit{and}\; 0 < \varepsilon < \varepsilon_1.$$

We write: $f(\boldsymbol{x};\varepsilon) = \varphi(\boldsymbol{x};\varepsilon) + o(\varphi(\boldsymbol{x};\varepsilon))$ uniformly in $\boldsymbol{x} \in \mathcal{D}$ as $\varepsilon \to 0$. Note that $\mathcal{D}$ may depend on ε.

Example 3.1 Let $\mathcal{D} = [0, 1]$ and $0 < \varepsilon < 1$. Then we have $\cos(\varepsilon x) = 1 + o(1)$ as $\varepsilon \to 0$ uniformly in $\mathcal{D}$, since for any given δ we can choose $\varepsilon_1 = \sqrt{\delta}$, such that $|\cos(\varepsilon x) - 1| \leqslant \varepsilon^2 x^2 \leqslant \varepsilon_1^2 = \delta$. $\square$

Example 3.2 Although $\cos(x/\varepsilon) = O(1)$ uniformly in $\boldsymbol{x} \in [0, 1]$ for $\varepsilon \to 0$, there is no constant K such that $\cos(x/\varepsilon) = K + o(1)$. $\square$

Example 3.3 $x + \sin(\varepsilon x) + \mathrm{e}^{-x/\varepsilon} = x + \varepsilon x + O(\varepsilon^3)$ as $\varepsilon \to 0$ for all $x \neq 0$, but not uniformly in $x \in [0, 1]$. More precisely, it is not uniformly in $x \in [\delta(\varepsilon), 1]$ for any $\delta = O(\varepsilon)$ and uniformly if $\varepsilon = o(\delta)$. If $x = O(\varepsilon)$, the otherwise exponentially small term is not small anymore. This is illustrated by the Figure 3.1. The difference between the original function and its non-uniform asymptotic approximation

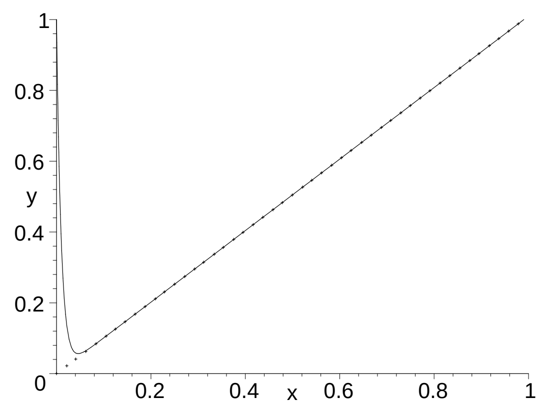


Figure 3.1: A plot of $x + \sin(\varepsilon x) + \mathrm{e}^{-x/\varepsilon}$ and its non-uniform asymptotic approximation $x + \varepsilon x$ for $\varepsilon = 0.01$.

is typically large in a neighbourhood of $x = 0$, while the size of this neighbourhood is $x = O(\varepsilon)$. This neighbourhood is an example of a boundary layer. The occurrence and behaviour of boundary layers will be discussed in more detail in Section 6. $\square$

3.1.2 Asymptotic expansions

Asymptotic approximations are usually structured in the form of a series expansion that helps us to construct an approximation systematically.

Definition 3.3 *The sequence $\{\mu_n(\varepsilon)\}_{n=0}^{\infty}$ is called an asymptotic sequence, if $\mu_{n+1}(\varepsilon) = o(\mu_n(\varepsilon))$, as $\varepsilon \to 0$, for each $n = 0, 1, 2, \cdots$.*

Example 3.4 Examples of asymptotic sequences (as $\varepsilon \to 0$) are

$$\mu_n(\varepsilon) = \varepsilon^n, \quad \mu_n(\varepsilon) = \varepsilon^{\frac{1}{2}n}, \quad \mu_n(\varepsilon) = \tan^n(\varepsilon), \quad \mu_n(\varepsilon) = \ln(\varepsilon)^{-n},$$
$$\mu_n(\varepsilon) = \varepsilon^p \ln(\varepsilon)^q \ \text{ where } \ p = 0, 1, 2..., \ \ q = 0...p \ \text{ and } \ n = \tfrac{1}{2}p(p+3) - q. \qquad \square$$

Definition 3.4 *If $\{\mu_n(\varepsilon)\}_{n=0}^{\infty}$ is an asymptotic sequence, then $f(\varepsilon)$ has an asymptotic expansion of N terms with respect to this sequence, denoted by*

$$f(\varepsilon) \sim \sum_{n=0}^{N-1} a_n \mu_n(\varepsilon),$$

where the coefficients a_n are independent of ε, if

$$f(\varepsilon) - \sum_{n=0}^{M} a_n \mu_n(\varepsilon) = o(\mu_M(\varepsilon)) \quad \text{as } \varepsilon \to 0$$

for each $M = 0, \ldots, N-1$. $\mu_n(\varepsilon)$ is called a gauge-function. If $\mu_n(\varepsilon) = \varepsilon^n$, we call the expansion an asymptotic power series.

Definition 3.5 *Two functions f and g are asymptotically equal up to N terms, with respect to the asymptotic sequence $\{\mu_n\}$, if $f - g = o(\mu_N)$. If the remaining error is clear from the context, this is sometimes denoted as $f \sim g$.*

Asymptotic expansions based on the same gauge functions may be added. They may be multiplied if the products of the gauge functions can be asymptotically ordered.

In contrast to ordinary series expansions, defined for an infinite number of terms, in asymptotic expansions only a finite (N) number of terms are considered. For $N \to \infty$ the series may either converge or diverge, but this is irrelevant for the asymptotic behaviour. In addition it may be worthwhile to note that it is not necessary for a convergent asymptotic expansion to converge to the expanded function.

For given $\{\mu_n(\varepsilon)\}_{n=0}^{\infty}$, the coefficients a_n can be determined uniquely by the following recursive procedure (provided μ_n are nonzero for ε near 0 and each of the limits below exist)

$$a_0 = \lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{\mu_0(\varepsilon)}, \quad a_1 = \lim_{\varepsilon\to 0} \frac{f(\varepsilon) - a_0\mu_0(\varepsilon)}{\mu_1(\varepsilon)}, \quad \ldots \quad a_{N-1} = \lim_{\varepsilon\to 0} \frac{f(\varepsilon) - \sum_{n=0}^{N-2} a_n\mu_n(\varepsilon)}{\mu_{N-1}(\varepsilon)}.$$

Example 3.5 A function may have different asymptotic expansions.

$$\begin{aligned} \tan(\varepsilon) &= \varepsilon + \tfrac{1}{3}\varepsilon^3 + \tfrac{2}{15}\varepsilon^5 + O(\varepsilon^7) \\ &= \sin\varepsilon + \tfrac{1}{2}(\sin\varepsilon)^3 + \tfrac{3}{8}(\sin\varepsilon)^5 + O((\sin\varepsilon)^7) \\ &= \varepsilon\cos\varepsilon + \tfrac{5}{6}(\varepsilon\cos\varepsilon)^3 + \tfrac{161}{120}(\varepsilon\cos\varepsilon)^5 + O((\varepsilon\cos\varepsilon)^7). \end{aligned}$$ □

Example 3.6 The following asymptotic expansion, related to the exponential integral Ei,

$$\varepsilon^{-1}\, \mathrm{e}^{-1/\varepsilon}\, \mathrm{Ei}(1/\varepsilon) = \sum_{n=0}^{N} n!\, \varepsilon^n + o(\varepsilon^N), \quad \text{where } \mathrm{Ei}(x) = \fint_{-\infty}^{x} \frac{\mathrm{e}^t}{t}\, \mathrm{d}t,$$

diverges as $N \to \infty$ if $\varepsilon \neq 0$. The accuracy increases with N, but on a smaller interval [49]. □

Example 3.7 Different functions may have the same asymptotic expansion.

$$\begin{aligned} \cos(\varepsilon) &= 1 - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{24}\varepsilon^4 + O(\varepsilon^6), \\ \cos(\varepsilon) + \mathrm{e}^{-1/\varepsilon} &= 1 - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{24}\varepsilon^4 + O(\varepsilon^6). \end{aligned}$$

Note that both asymptotic expansions, considered as regular power series in ε, converge to $\cos(\varepsilon)$ rather than $\cos(\varepsilon) + \mathrm{e}^{-1/\varepsilon}$. □

Theorem 3.8 *An asymptotic expansion vanishes only if the coefficients vanish, i.e.*

$$\big\{a_0\mu_0(\varepsilon)+a_1\mu_1(\varepsilon)+a_2\mu_2(\varepsilon)+\ldots=0 \quad (\varepsilon\to 0)\big\} \;\Leftrightarrow\; \big\{a_0=a_1=a_2=\ldots=0\big\}.$$

Proof

The sequence $\{\mu_n\}$ is asymptotically ordered, so both $\mu_0 a_0 = -\mu_1 a_1 - \ldots = O(\mu_1)$ and $\mu_1 = o(\mu_0)$. So there is a positive constant K such that for any positive δ there is an ε-interval where $|a_0\mu_0| < \delta K|\mu_0|$, which is only possible if $a_0 = 0$. This may now be repeated for a_1, *etc.* This proves $\Rightarrow$. The proof of $\Leftarrow$ is trivial. □

3.2 Basic definitions and theorems

1. ***O*** (Big O)
   $f(\varepsilon) = O(\varphi(\varepsilon))$ as $\varepsilon \to 0$ if there are a fixed constant $K > 0$ and an interval $(0, \varepsilon_1)$ such that

   $$|f(\varepsilon)| \leqslant K|\varphi(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_1.$$

   Intuitive interpretation: f can be embraced completely by $|\varphi|$ (up to a multiplicative constant) in a neighbourhood of 0. A crude estimate (for example $\sin\varepsilon = O(1/\varepsilon)$) is not incorrect, but a sharp estimate is more informative.

   Examples: $\sin\varepsilon = O(\varepsilon)$, $(1-\varepsilon)^{-1} = O(1)$, $\sin(1/\varepsilon) = O(1)$, $(\varepsilon+\varepsilon^2)^{-1} = O(\varepsilon^{-1})$, $\ln((1+\varepsilon)/\varepsilon) = O(\ln\varepsilon)$.

2. ***o*** (small o)
   $f(\varepsilon) = o(\varphi(\varepsilon))$ as $\varepsilon \to 0$ if for every $\delta > 0$ there is an interval $(0, \varepsilon_1)$ such that

   $$|f(\varepsilon)| \leqslant \delta|\varphi(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_1.$$

   Intuitive interpretation: f is always smaller than any multiple (however small) of $|\varphi|$ in a neighbourhood of 0. Again, a crude estimate is not incorrect, but a sharp estimate is more informative.

   Examples: $\sin(2\varepsilon) = o(1)$, $\cos\varepsilon = o(\varepsilon^{-1})$, $\mathrm{e}^{-a/\varepsilon} = o(\varepsilon^n)$ for any $a > 0$ and any n.

3. ***O*$_s$** (sharp O)
   $f(\varepsilon) = O_s(\varphi(\varepsilon))$ as $\varepsilon \to 0$ if $f(\varepsilon) = O(\varphi(\varepsilon))$ and $f(\varepsilon) \neq o(\varphi(\varepsilon))$.

   Intuitive interpretation: f behaves exactly the same (up to a multiplicative constant) as φ in a neighbourhood of 0.

   Examples: $2\sin\varepsilon = O_s(\varepsilon)$, $3\cos\varepsilon = O_s(1)$, but there is *no n* such that $\ln\varepsilon = O_s(\varepsilon^n)$.

4. **Similar behaviour.**

   (i) If $f = o(\varphi)$ then $f = O(\varphi)$.

   (ii) If $\displaystyle\lim_{\varepsilon\downarrow 0}\left|\frac{f(\varepsilon)}{\varphi(\varepsilon)}\right| = 0$ then $f = o(\varphi)$.

   (iii) If $\displaystyle\lim_{\varepsilon\downarrow 0}\left|\frac{f(\varepsilon)}{\varphi(\varepsilon)}\right| = L \in [0,\infty)$ then $f = O(\varphi)$.

   (iv) If $\displaystyle\lim_{\varepsilon\downarrow 0}\left|\frac{f(\varepsilon)}{\varphi(\varepsilon)}\right| = L \in (0,\infty)$ then $f = O_s(\varphi)$.

(v) If $f = O(\varphi)$ and $\varphi = O(f)$ then $f = O_s(\varphi)$.

The reverse is certainly not true: (i) $\sin\varepsilon = O(\varepsilon)$ but $\sin\varepsilon \neq o(\varepsilon)$. (ii) If $f \equiv 0$ and $\varphi \equiv 0$ on an interval containing $\varepsilon = 0$, then $f = o(\varphi)$ but $\lim\limits_{\varepsilon\downarrow 0} |f/\varphi|$ does not exist. (iii,iv) $\varepsilon\sin(1/\varepsilon) = O_s(\varepsilon)$, but $\lim\limits_{\varepsilon\downarrow 0} |\sin(1/\varepsilon)|$ does not exist. (v) $\sin(1/\varepsilon) = O_s(1)$ but $1 \neq O(\sin(1/\varepsilon))$.

5. **Asymptotic approximation.**
$\varphi(\varepsilon)$ is an asymptotic approximation to $f(\varepsilon)$ as $\varepsilon \to 0$, denoted by $f \sim \varphi$, if

$$f(\varepsilon) = \varphi(\varepsilon) + o(\varphi(\varepsilon)) \quad \text{as } \varepsilon \to 0,$$

Intuitive interpretation: If $\lim\limits_{\varepsilon\to 0} f/\varphi = 1$ then $f \sim \varphi$. *Note:* $f \sim 0$ is only possible if $f \equiv 0$.
Examples: $\sin\varepsilon \sim \varepsilon$, $(\varepsilon + \varepsilon^2)^{-1} \sim 1/\varepsilon$, $\ln(a\varepsilon) \sim \ln\varepsilon$ for any $a > 0$.

6. **Pointwise asymptotic approximation.**
$\varphi(x,\varepsilon)$ is a pointwise asymptotic approximation to $f(x,\varepsilon)$ as $\varepsilon \to 0$ if

$$f(x,\varepsilon) \sim \varphi(x,\varepsilon) \quad \text{for fixed } x.$$

Intuitive interpretation: $f(x,\varepsilon)$ is approximated asymptotically better and better by $\varphi(x,\varepsilon)$ for $\varepsilon \to 0$ and x *fixed*. We don't know anything yet if we allow x to become small or large (within the domain).

Examples: $\sin(x+\varepsilon) \sim \sin x$ and $\sin x \neq 0$, $1/(\varepsilon + x) \sim 1/x$ and $x \neq 0$. Note that in the last example the approximation fails if we would scale $x = \varepsilon^n t$ for any $n \geqslant 1$.

7. **Uniform asymptotic approximation.**
The continuous function $\varphi(x,\varepsilon)$ is a uniform asymptotic approximation to the continuous function $f(x,\varepsilon)$ for $x \in \mathcal{D}$ as $\varepsilon \to 0$, if the way φ approaches f is the same for all x.
More precisely: if for any positive number δ there is an ε_1 (independent of x and ε) such that

$$|f(x,\varepsilon) - \varphi(x,\varepsilon)| \leqslant \delta|\varphi(x,\varepsilon)| \quad \text{for } x \in \mathcal{D} \quad \text{and } 0 < \varepsilon < \varepsilon_1.$$

Intuitive interpretation:
$f(x,\varepsilon)$ is approximated uniformly by $\varphi(x,\varepsilon)$, if the approximation is preserved with any scaling of $x = a(\varepsilon) + b(\varepsilon)t$, valid in the domain of f. In formulas (with a scaling $x = \varepsilon t \in [0, K]$ as an example):

$$\text{if } f(x,\varepsilon) \sim \varphi(x,\varepsilon) \text{ and } \varphi(\varepsilon t,\varepsilon) \sim g(t,\varepsilon), \text{ then also } f(\varepsilon t,\varepsilon) \sim g(t,\varepsilon).$$

Examples:

$\cos(\varepsilon) + \mathrm{e}^{-x/\varepsilon} \sim 1$	only pointwise for $x \in (0,\infty)$. Not uniform: take $x = \varepsilon t$.
$\cos(\varepsilon) + \mathrm{e}^{-x/\varepsilon} \sim 1$	pointwise and uniformly for $x \in [a,\infty)$, $a > 0$.
$\cos(\varepsilon) + \mathrm{e}^{-t} \sim 1 + \mathrm{e}^{-t}$	uniformly for $t \in [0,\infty)$.
$\sin(\varepsilon x + \varepsilon) \sim \varepsilon(x+1)$	only pointwise for $x \in (-\infty,\infty)$. Take $x = t/\varepsilon$.
$\sin(\varepsilon x + \varepsilon) \sim \varepsilon(x+1)$	uniformly for $x \in [-a,a]$, $0 < a < \infty$.
$2 + \sin(t + \varepsilon) \sim 2 + \sin(t)$	uniformly for $t \in (-\infty,\infty)$.
$2 + \mathrm{e}^{\varepsilon}\sin(t + \varepsilon t) \sim 2 + \sin(t)$	only pointwise for $t \in \mathbb{R}$. Note that $\sin(t+\varepsilon t) = \sin t + O(\varepsilon t)$.
$2 + \mathrm{e}^{\varepsilon}\sin(\tau) \sim 2 + \sin(\tau)$	uniform for $\tau \in \mathbb{R}$. Note that we rescaled $\tau = (1+\varepsilon)t$.

Uniform implies *pointwise*, but the reverse is not necessarily true. See the above examples.

8. If f and φ are absolutely integrable, and $f(x, \varepsilon) \sim \varphi(x, \varepsilon)$ uniformly on a domain $\mathcal{D}$, while $\int_{\mathcal{D}} |\varphi| \mathrm{d}x = O(\int_{\mathcal{D}} \varphi \mathrm{d}x)$, then $\int_{\mathcal{D}} f(x, \varepsilon)\, \mathrm{d}x \sim \int_{\mathcal{D}} \varphi(x, \varepsilon)\, \mathrm{d}x$.

9. **Asymptotic sequence.**
   The sequence $\{\mu_n(\varepsilon)\}$ is called an asymptotic sequence, if $\mu_{n+1} = o(\mu_n)$ as $\varepsilon \to 0$ for each $n = 0, 1, 2, \cdots$. This is denoted symbolically

$$\mu_0 \gg \mu_1 \gg \mu_2 \gg \cdots \gg \mu_n \gg \ldots$$

   Common examples are $\mu_n = \varepsilon^n$, or more generally $\mu_n = \delta(\varepsilon)^n$ if $\delta(\varepsilon) = o(1)$. Combinations of ε and $\ln(\varepsilon)$ yield the sequence $\mu_{n,k} = \varepsilon^n \ln(\varepsilon)^k$, where $k = n, \cdots, 0$ and

$$\ln \varepsilon \gg 1 \gg \varepsilon \ln(\varepsilon) \gg \varepsilon \gg \varepsilon^2 \ln(\varepsilon)^2 \gg \varepsilon^2 \ln(\varepsilon) \gg \varepsilon^2 \gg \ldots$$

10. **Asymptotic expansion.**
    If $\{\mu_n(\varepsilon)\}$ is an asymptotic sequence, then $f(\varepsilon)$ has an asymptotic expansion of $N + 1$ terms with respect to this sequence, denoted by

$$f(\varepsilon) \sim \sum_{n=0}^{N} a_n \mu_n(\varepsilon),$$

    where the coefficients a_n are independent of ε, if for each $M = 0, \ldots, N$

$$f(\varepsilon) - \sum_{n=0}^{M} a_n \mu_n(\varepsilon) = o(\mu_M(\varepsilon)) \quad \text{as} \;\; \varepsilon \to 0.$$

    $\mu_n(\varepsilon)$ is called a **gauge function** or **order function**.

    If $\mu_n(\varepsilon) = \varepsilon^n$, we call the expansion an asymptotic power series. Any Taylor series in ε around $\varepsilon = 0$ is also an asymptotic power series.

    Asymptotic expansions, based on Taylor expansions in ε^n, of elementary functions:

$$\begin{aligned}
\mathrm{e}^{\varepsilon} &= 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + \ldots \\
\sin(\varepsilon) &= \varepsilon - \tfrac{1}{6}\varepsilon^3 + \ldots \\
\cos(\varepsilon) &= 1 - \tfrac{1}{2}\varepsilon^2 + \ldots \\
\frac{1}{1-\varepsilon} &= 1 + \varepsilon + \varepsilon^2 + \ldots \\
\ln(1-\varepsilon) &= -\varepsilon - \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{3}\varepsilon^3 - \ldots \\
\ln(1+\varepsilon) &= \varepsilon - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{3}\varepsilon^3 - \ldots \\
(1+\varepsilon)^{\alpha} &= 1 + \alpha\varepsilon + \tfrac{1}{2}\alpha(\alpha-1)\varepsilon^2 + \ldots
\end{aligned}$$

    Examples of combinations (which are sometimes not Taylor expansions in ε^n)

$$\begin{aligned}
\varepsilon^{\varepsilon} = \mathrm{e}^{\varepsilon \ln \varepsilon} &= 1 + \varepsilon \ln \varepsilon + \tfrac{1}{2}\varepsilon^2 (\ln \varepsilon)^2 + \ldots \\
\ln(\sin \varepsilon) &= \ln \varepsilon - \tfrac{1}{6}\varepsilon^2 + \ldots \\
\ln(\cos \varepsilon) &= -\tfrac{1}{2}\varepsilon^2 - \tfrac{1}{12}\varepsilon^4 + \ldots \\
\frac{1}{1 - f(\varepsilon)} &= 1 + f(\varepsilon) + f(\varepsilon)^2 + \ldots, \qquad \text{if} \;\; f(\varepsilon) = o(1).
\end{aligned}$$

11. **How to determine the coefficients.**
The coefficients a_n of an asymptotic expansion can be determined uniquely (for given $\mu_n(\varepsilon)$) by the following recursive procedure

$$a_0 = \lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{\mu_0(\varepsilon)}, \quad a_1 = \lim_{\varepsilon\to 0} \frac{f(\varepsilon) - a_0\mu_0(\varepsilon)}{\mu_1(\varepsilon)}, \; \dots \; a_N = \lim_{\varepsilon\to 0} \frac{f(\varepsilon) - \sum_{n=0}^{N-1} a_n\mu_n(\varepsilon)}{\mu_N(\varepsilon)},$$

provided μ_n are nonzero for ε near 0 and each of the limits exist.

12. **Convergent and asymptotic.**
Let $\{\mu_n(\varepsilon)\}$ be an asymptotic sequence, with $\mu_0 = 1$ and $\varepsilon > 0$, and let

$$f(\varepsilon) = \sum_{n=0}^{N} a_n\mu_n(\varepsilon) + R_N(\varepsilon).$$

If the series converges for $N \to \infty$, then $\lim_{N\to\infty} R_N(\varepsilon) = 0$. If the series is an asymptotic expansion for $\varepsilon \to 0$, then $\lim_{\varepsilon\to 0} R_N(\varepsilon) = 0$. A convergent power series (like a Taylor series) is also an asymptotic expansion. An asymptotic expansion is not necessarily convergent.

13. **Asymptotically equal**
Two functions f and g are asymptotically equal up to N terms, with respect to the asymptotic sequence $\{\mu_n\}$, if $f - g = o(\mu_N)$.

14. **The fundamental theorem of asymptotic expansions [10]**
An asymptotic expansion vanishes if and only if the coefficients vanish, i.e.

$$\Big\{a_0\mu_0(\varepsilon) + a_1\mu_1(\varepsilon) + a_2\mu_2(\varepsilon) + \ldots = 0 \quad (\varepsilon \to 0)\Big\} \;\Leftrightarrow\; \Big\{a_0 = a_1 = a_2 = \ldots = 0\Big\}.$$

15. **Poincaré expansion.**
Let $\{\mu_n(\varepsilon)\}$ be an asymptotic sequence of **order functions**. If $f(x,\varepsilon)$ has an asymptotic expansion with respect to this sequence, given by

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\mu_n(\varepsilon),$$

where the **shape functions** $a_n(x)$ are **independent** of ε, then this expansion is called a Poincaré expansion. *Note*: a Poincaré expansion is *never* Poincaré anymore after (nontrivial) rescaling x.

16. **Regular and singular expansion.**
If a Poincaré expansion is uniform in x on a given domain $\mathcal{D}$ this expansion is called a regular expansion. Else, the expansion is called a singular expansion. *Examples*: $\sin(x+\varepsilon) \sim \sin x + \varepsilon\cos x + O(\varepsilon^2)$ is uniform on $\mathbb{R}$. $(x+\varepsilon)^{-1} = x^{-1} - \varepsilon x^{-2} + O(\varepsilon^2)$ is uniform on $[A,\infty)$ for any $A > 0$.

Note: A typical indication for non-uniformity is a scaling, such that the asymptotic ordering of the terms is violated. In other words, a scaled $x = x(\varepsilon)$ with $a_1(x)\mu_1(\varepsilon) \not\ll a_0(x)\mu_0(\varepsilon)$, etc.

17. **Role of scaling.**
A Poincaré expansion and its region of uniformity depends (among other things) on the chosen scaling $x = x_0 + \delta(\varepsilon)\xi$ and the domain $\mathcal{D}$.

For example, $\mathrm{e}^{-x/\varepsilon} + \sin(x+\varepsilon) = \sin(x) + O(\varepsilon)$ is regular on any positive interval $[a, b]$ with $a, b = O(1)$ but is singular on $(0, b]$, while $\mathrm{e}^{-t} + \sin(\varepsilon t + \varepsilon) = \mathrm{e}^{-t} + \varepsilon(t+1) + O(\varepsilon^3)$ is regular on any finite fixed interval.

18. **Manipulations of asymptotic expansions.**
Let $f(x, \varepsilon)$ and $g(x, \varepsilon)$ have Poincaré expansions on $\mathcal{D}$ with asymptotic sequence $\{\mu_n(\varepsilon)\}$

$$f(x, \varepsilon) = \mu_0(\varepsilon)a_0(x) + \mu_1(\varepsilon)a_1(x) + \cdots$$
$$g(x, \varepsilon) = \mu_0(\varepsilon)b_0(x) + \mu_1(\varepsilon)b_1(x) + \cdots$$

Addition. Then the sum has the following asymptotic expansion

$$f + g = \mu_0(a_0 + b_0) + \mu_1(a_1 + b_1) + \cdots$$

Multiplication. If $\{\mu_k\mu_n\}$ can be asymptotically ordered to the asymptotic sequence $\{\gamma_n\}$, with $\gamma_0 = \mu_0^2$, $\gamma_1 = \mu_0\mu_1$, $\gamma_2 = O(\mu_0\mu_2 + \mu_1^2)$, etc., then the product has the asymptotic expansion

$$fg = (\mu_0 a_0 + \mu_1 a_1 + \cdots)(\mu_0 b_0 + \mu_1 b_1 + \cdots) = \gamma_0 a_0 b_0 + \gamma_1(a_0 b_1 + a_1 b_0) + \gamma_2(\cdots) + \cdots$$

Integration. If the approximation is uniform, f, a_0, a_1, etc. are absolute-integrable on $\mathcal{D}$, while $\int_{\mathcal{D}} a_n \,\mathrm{d}x \neq 0$, then we can integrate term by term and obtain the asymptotic expansion

$$\int_{\mathcal{D}} f(x, \varepsilon)\,\mathrm{d}x = \mu_0 \int_{\mathcal{D}} a_0(x)\,\mathrm{d}x + \mu_1 \int_{\mathcal{D}} a_1(x)\,\mathrm{d}x + \cdots$$

Differentiation. This is the least obvious. Consider the counter example

$$f(x, \varepsilon) = \tfrac{1}{2}x^2 + \varepsilon\cos\left(\tfrac{x}{\varepsilon}\right) = \tfrac{1}{2}x^2 + O(\varepsilon), \quad \text{but} \quad f'(x, \varepsilon) = x - \sin\left(\tfrac{x}{\varepsilon}\right) \neq x + O(\varepsilon).$$

However, if both f and f' have asymptotic expansions with asymptotic sequence $\{\mu_n(\varepsilon)\}$, say

$$f(x, \varepsilon) = \mu_0(\varepsilon)a_0(x) + \mu_1(\varepsilon)a_1(x) + \cdots, \qquad f'(x, \varepsilon) = \mu_0(\varepsilon)q_0(x) + \mu_1(\varepsilon)q_1(x) + \cdots$$

then the derivative of the expansion of f is the expansion of derivative f', and satisfy

$$a_0' = q_0, \quad a_1' = q_1, \quad \text{etc.}$$

3.3 Asymptotic Expansions: Applications

3.3.1 General procedure for algebraic equations

The existence of an asymptotic expansion yields a class of methods to solve problems that depend on a parameter which is typically small in the range of interest. Such methods are called *perturbation methods*. The importance of these methods are two-fold. They provide analytic solutions to otherwise intractable problems, and the asymptotic structure of the solution provides instant insight into the dominating qualities.

If $x(\varepsilon)$ is implicitly given as the solution of an algebraic equation

$$\mathcal{F}(x,\varepsilon) = 0 \tag{3.1}$$

we may solve this asymptotically for $\varepsilon \to 0$ in the following steps.

(i) First we prove, make plausible, or check in one way or another that a solution *exists*, and try to find out if this solution is unique or there are more. This is not really an asymptotic question, but important because the approximations involved later in the solution process may fool us: the approximated equation may have no solutions while the original has, or the other way round. Sometimes the existence of solutions is obvious straightaway, but sometimes global arguments should be invoked.

(ii) Then we have to find the order of magnitude of the sought solution, say $x(\varepsilon) = \gamma(\varepsilon)X(\varepsilon)$ with $X = O_s(1)$. Unless we have scaled the problem already correctly, the solution is not necessarily $O(1)$. Often, we cannot decide with certainty, and we have to make a suitable assumption that is consistent with all the information we have, and proceed to construct successfully a solution or until we encounter a contradiction.

Another point of concern is the fact that there may be more solutions with different scalings.

The scaling function $\gamma(\varepsilon)$ is found such that it yields a meaningful $X = O_s(1)$ in the limit $\varepsilon \to 0$. This is called a *distinguished limit*, while the reduced equation for $X(0)$, *i.e.* $\mathcal{F}_0(X) = 0$, is called a *significant degeneration* (there may be more than one.) We can rescale $\mathcal{F}$ and x such $\mathcal{F}(x,\varepsilon) = 0$ becomes $\mathcal{G}(X,\varepsilon) = 0$ while $\mathcal{G}(X,0) = O(1)$.

(iii) The final stage is to make an assumption about an asymptotic expansion of the solution X for small ε

$$X(\varepsilon) = X_0 + \mu_1(\varepsilon)X_1 + \mu_2(\varepsilon)X_2 + \ldots$$

This is *only* an assumption, based on a successful and consistent construction later. If we encounter a contradiction, we have to go back and correct or alter the assumed expansion.

If both $X(\varepsilon)$ and $\mathcal{G}(X,\varepsilon)$ have an asymptotic series expansion with the same gauge functions, $X(\varepsilon)$ may be determined asymptotically by the following perturbation method. We expand X, substitute this expansion in $\mathcal{G}$, and expand $\mathcal{G}$ to obtain

$$\mathcal{G}(X,\varepsilon) = \mathcal{G}_0(X_0) + \mu_1(\varepsilon)\mathcal{G}_1(X_1,X_0) + \mu_2(\varepsilon)\mathcal{G}_2(X_2,X_1,X_0) + \ldots = 0.$$

From the Fundamental Theorem of asymptotic expansions (3.3.8) it follows that each term $\mathcal{G}_n$ vanishes, and the sequence of coefficients (X_n) can be determined by induction:

$$\mathcal{G}_0(X_0) = 0, \qquad \mathcal{G}_1(X_1,X_0) = 0, \qquad \mathcal{G}_2(X_2,X_1,X_0) = 0, \qquad \textit{etc.} \tag{3.2}$$

It should be noted that finding the sequence of gauge functions (μ_n) is of particular importance. This is in general done iteratively, but sometimes a good guess also works. For example, if $\mathcal{G}$ is a smooth function of ε, in particular in $\varepsilon = 0$, then in most cases an asymptotic power series will work, *i.e.* $\mu_n(\varepsilon) = \varepsilon^n$.

We have to realise that a successful construction is not a *proof* for its correctness. Strictly mathematical proofs are usually very difficult, and in the context of modelling not common. Successfully finding a consistent solution is normally the strongest indication for its correctness we can obtain.

3.3.2 Example: roots of a polynomial

We illustrate this procedure by the following example. Consider the roots for $\varepsilon \to 0$ of the equation

$$x^3 - \varepsilon x^2 + 2\varepsilon^3 x + 2\varepsilon^6 = 0.$$

Since the polynomial is of 3^{d} order, and is negative for $x = -1$ (and ε small), positive in $x = \varepsilon^2$, negative in $x = -\frac{1}{2}\varepsilon$, and positive in $x = 1$, there are exactly 3 real solutions $x^{(1)}, x^{(2)}, x^{(3)}$.

From the structure of the equation it seems reasonable to assume that the order of magnitude of the solutions scale like a power of ε. We write

$$x = \varepsilon^n X(\varepsilon), \quad X = O_s(1)$$

We have to determine exponent n first. This is done by balancing terms, and then seek such n that produce a non-trivial limit under the limit $\varepsilon \to 0$: the *distinguished limits* of step (ii) above.
We compare asymptotically the coefficients in the equation that remain after scaling

$$\varepsilon^{3n} X^3 - \varepsilon^{1+2n} X^2 + 2\varepsilon^{3+n} X + 2\varepsilon^6 = 0.$$

Consider now the order of magnitude of the coefficients:

$$\varepsilon^{3n}, \quad \varepsilon^{1+2n}, \quad \varepsilon^{3+n}, \quad \varepsilon^6.$$

By dividing by the biggest coefficient (this depends on n), we can always make sure that one coefficient is 1 and the others are smaller. For example, if $n = 0$ we have

$$1, \quad \varepsilon, \quad \varepsilon^3, \quad \varepsilon^6.$$

If $n = 2$ we have

$$\varepsilon, \quad 1, \quad 1, \quad \varepsilon.$$

If $n = 4$ we have

$$\varepsilon^6, \quad \varepsilon^3, \quad \varepsilon, \quad 1.$$

If none balance (like for $n = 0$ and $n = 4$), the asymptotically biggest, with coefficient 1, would be zero on its own, which thus implies to leading order that $X = 0$. However, this is *not* $O_s(1)$ and therefore not a valid scaling. So at least two should be of the same order of magnitude *and* dominate (like with $n = 2$).

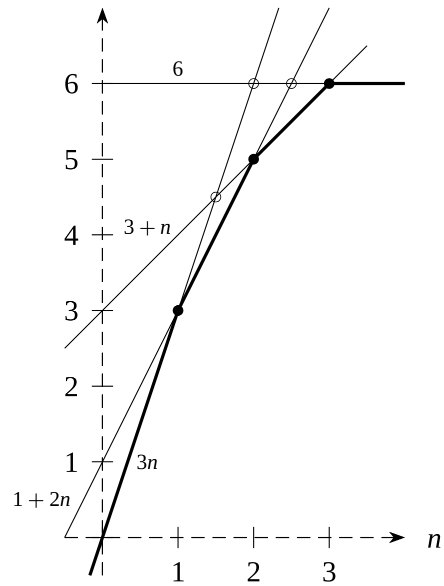


Figure 3.2: Analysis of distinguished limits.

In other words: in order to have a meaningful (or "significant") degenerate solution $X(0) = O_s(1)$, at least two terms of the equation should be asymptotically equivalent, and at the same time of leading order when $\varepsilon \to 0$.

So this leaves us with the task to compare the exponents $3n$, $1+2n$, $3+n$, 6 as a function of n. Consider the Figure 3.2. The solid lines denote the exponents of the powers of ε, that occur in the coefficients of the equation considered. At the intersections of these lines, denoted by the open and closed circles, we find the candidates of distinguished limits, *i.e.* the points where at least two coefficients are asymptotically equivalent. Finally, only the closed circles are the distinguished limits, because these are located along the lower envelope (thick solid line) and therefore correspond to leading order terms when $\varepsilon \to 0$. We have now three cases.

$n = 1$.

$$\varepsilon^3 X^3 - \varepsilon^3 X^2 + 2\varepsilon^4 X + 2\varepsilon^6 = 0, \quad \text{or} \quad X^3 - X^2 + 2\varepsilon X + 2\varepsilon^3 = 0.$$

From the structure of the equation it seems reasonable to assume that X has an asymptotic expansion in powers of ε. If we assume the expansion $X = X_0 + \varepsilon X_1 + \ldots$, we finally have

$$X_0^3 - X_0^2 = 0, \quad 3X_0^2 X_1 - 2X_0 X_1 + 2X_0 = 0, \quad \textit{etc.}$$

and so $X_0 = 1$, and $X_1 = -2$, *etc.* leading to $x(\varepsilon) = \varepsilon - 2\varepsilon^2 + \ldots$ Note that solution $X_0 = 0$ is excluded because that would change the order of the scaling!

$n = 2$.

$$\varepsilon^6 X^3 - \varepsilon^5 X^2 + 2\varepsilon^5 X + 2\varepsilon^6 = 0, \quad \text{or} \quad \varepsilon X^3 - X^2 + 2X + 2\varepsilon = 0.$$

From the structure of the equation it seems reasonable to assume that X has an asymptotic expansion in powers of ε. If we assume the expansion $X = X_0 + \varepsilon X_1 + \ldots$, we finally have

$$-X_0^2 + 2X_0 = 0, \quad X_0^3 - 2X_0 X_1 + 2X_1 + 2 = 0, \quad \textit{etc.}$$

and so $X_0 = 2$, $X_1 = 5$, *etc.* , leading to $x(\varepsilon) = 2\varepsilon^2 + 5\varepsilon^3 + \ldots$

$n = 3$.

$$\varepsilon^9 X^3 - \varepsilon^7 X^2 + 2\varepsilon^6 X + 2\varepsilon^6 = 0, \quad \text{or} \quad \varepsilon^3 X^3 - \varepsilon X^2 + 2X + 2 = 0.$$

From the structure of the equation it seems reasonable to assume that X has an asymptotic expansion in powers of ε. If we assume the expansion $X = X_0 + \varepsilon X_1 + \ldots$, we finally have

$$2X_0 + 2 = 0, \quad -X_0^3 + 2X_1 = 0, \quad \textit{etc.}$$

and so $X_0 = -1$, $X_1 = -\frac{1}{2}$, *etc.* , leading to $x(\varepsilon) = -\varepsilon^3 - \frac{1}{2}\varepsilon^4 + \ldots$ □

It is not always so easy to guess the general form of the gauge functions. Then all terms have to be estimated iteratively by a similar process of balancing as for the leading order term. See the exercises.

3.4 Asymptotic Expansions: Assignments

3.4.1 Asymptotic order

3.4.1.1

Prove, for functions in $\varepsilon \downarrow 0$, that

(a) If $f = O(\phi)$ and $g = o(\psi)$, then $fg = o(\phi\psi)$.

(b) If $f = O(\phi)$ and $g = o(\phi)$, then $f + g = O(\phi)$.

(c) If $f = O(\phi)$ and $\phi = o(\psi)$, then $f = o(\psi)$.

(d) If $f = o(\phi)$ and $\phi = O(\psi)$, then $f = o(\psi)$.

(e) If $f = O(\phi)$ and $\phi = O(f)$, then $f = O_s(\phi)$.

3.4.2 Asymptotic expansions in ε

3.4.2.1

What values of α, if any, yield (i) $f = \mathcal{O}(\varepsilon^\alpha)$, (ii) $f = o(\varepsilon^\alpha)$, (iii) $f = \mathcal{O}_s(\varepsilon^\alpha)$ as $\varepsilon \to 0$?

(a) $f = \sqrt{1+\varepsilon^2}$

(b) $f = \varepsilon \sin(\varepsilon)$

(c) $f = (1 - \mathrm{e}^\varepsilon)^{-1}$

(d) $f = \ln(1+\varepsilon)$

(e) $f = \varepsilon \ln(\varepsilon)$

(f) $f = \sin(1/\varepsilon)$

(g) $f = \sqrt{x+\varepsilon}$, where $0 \leqslant x \leqslant 1$

(h) $f = \mathrm{e}^{-x/\varepsilon}$, where $x \geqslant 0$

3.4.2.2

Determine asymptotic expansions for $\varepsilon \to 0$ with respect to $\{\varepsilon^n (\ln \varepsilon)^k\}$ of

(a) $\varepsilon / \tan \varepsilon$,

(b) $\varepsilon / (1 - \varepsilon^\varepsilon)$,

(c) $1/\ln(\sin \varepsilon)$,

(d) $(1 - \varepsilon + \varepsilon^2 \ln \varepsilon)/(1 - \varepsilon \ln \varepsilon - \varepsilon + \varepsilon^2 \ln \varepsilon)$.

3.4.2.3

Assuming $f \sim a\varepsilon^{\alpha} + b\varepsilon^{\beta} + \ldots$, find α, β (with $\alpha < \beta$) and nonzero a, b for the following functions:

(a) $f = 1/(1 - \mathrm{e}^{\varepsilon})$

(b) $f = \sinh(\sqrt{1 + \varepsilon x})$ for $0 < x < \infty$.

(c) $f = \int_0^{\varepsilon} \sin(x + \varepsilon x^2)\,\mathrm{d}x$

3.4.3 Asymptotic sequences

3.4.3.1

Are the following sequences *asymptotic sequences* for $\varepsilon \to 0$. If not, arrange them so that they are or explain why it is not possible to do so.

(a) $\phi_n = (1 - \mathrm{e}^{-\varepsilon})^n$ for $n = 0, 1, 2, 3, \ldots$

(b) $\phi_n = [2\sinh(\varepsilon/2)]^{n/2}$ for $n = 0, 1, 2, 3, \ldots$

(c) $\phi_n = 1/\varepsilon^{1/n}$ for $n = 1, 2, 3, \ldots$

(d) $\phi_1 = 1, \phi_2 = \varepsilon, \phi_3 = \varepsilon^2, \phi_4 = \varepsilon\ln(\varepsilon), \phi_5 = \varepsilon^2\ln(\varepsilon), \phi_6 = \varepsilon\ln^2(\varepsilon), \phi_7 = \varepsilon^2\ln^2(\varepsilon)$.

(e) $\phi_n = \varepsilon^{n\varepsilon}$ for $n = 0, 1, 2, 3, \ldots$

(f) $\phi_n = \varepsilon^{n/\varepsilon}$ for $n = 0, 1, 2, 3, \ldots$

3.4.4 Asymptotic expansions in x and ε

3.4.4.1

Find a one-term asymptotic approximation, for $\varepsilon \to 0$, of the form $f(x,\varepsilon) \sim \phi(x)$ that holds for $-1 < x < 1$. Sketch $f(x,\varepsilon)$ and ϕ, and then explain why the approximation is not uniform for $-1 < x < 1$.

(a) $f(x,\varepsilon) = x + \exp\big((x^2 - 1)/\varepsilon\big)$

(b) $f(x,\varepsilon) = x + \tanh(x/\varepsilon)$

(c) $f(x,\varepsilon) = x + 1/\cosh(x/\varepsilon)$

3.4.4.2

Determine, if possible, uniform asymptotic expansions for $\varepsilon \to 0$ and $x \in [0, 1]$ of

(a) $\sin(\varepsilon x)$,

(b) $1/(\varepsilon + x)$,

(c) $x \log(\varepsilon x)$,

(d) $\mathrm{e}^{-\sin(x)\varepsilon}$,

(e) $\mathrm{e}^{-\sin(x)/\varepsilon}$.

(f) $2\log(1 + x)/(x^2 + \varepsilon^2)$.

3.4.5 Solving algebraic equations asymptotically

3.4.5.1

Find a two-term asymptotic expansion, for $\varepsilon \to 0$, of each solution x of the following equations.

(a) $\varepsilon x^3 - 3x + 1 = 0$,

(b) $\varepsilon x^3 - x + 2 = 0$,

(c) $x^{2+\varepsilon} = 1/(x + 2\varepsilon)$, $(x > 0)$.

(d) $x^2 - 1 + \varepsilon \tanh(x/\varepsilon) = 0$

(e) $x = a + \varepsilon x^k$ for $x > 0$. Consider $0 < k < 1$ and $k > 1$.

(f) $1 - 2x + x^2 - \varepsilon x^3 = 0$.

3.4.5.2

Derive step by step, by iteratively scaling $x(\varepsilon) = \mu_0(\varepsilon)x_0 + \mu_1(\varepsilon)x_1 + \mu_2(\varepsilon)x_2 + \dots$ and balancing, that a third order asymptotic solution (for $\varepsilon \to 0$) of the equation

$$\ln(\varepsilon x) + x = a,$$

is given by

$$x(\varepsilon) = \ln \varepsilon^{-1} - \ln\big(\ln \varepsilon^{-1}\big) + a + o(1).$$

Find a more efficient expansion based on an alternative asymptotic sequence of gauge functions by combining $\mathrm{e}^{-a}\,\varepsilon$.

3.4.5.3

Analyse asymptotically for $\varepsilon \to 0$ the zeros of $\mathrm{e}^{-x/\varepsilon^2} + x - \varepsilon$.

3.4.5.4

Solve asymptotically, for large n, the n-th positive solution $x = x_n$ of

$$x = \tan x.$$

Hint: for large n and $x_n > 0$, $x_n = \tan(x_n)$ is large, and so x_n must be near (in fact: just before) a pole of tan. *If we count the trivial first solution as $x_0 = 0$, then $x_n \simeq (n + \frac{1}{2})\pi$. Write $\varepsilon^{-1} = (n + \frac{1}{2})\pi$, and $x_n = \varepsilon^{-1} - y(\varepsilon)$ with $0 < y < \frac{1}{2}\pi$ such that $\tan(x) = \cot(y)$. Solve asymptotically for small ε.*

Generalise this result to the solutions of

$$x = \alpha \tan x$$

for $\alpha > 0$. Note that solution x_1 seems lost for $\alpha > 1$. Do you see where it disappeared to?

3.4.5.5 The pivoted barrier

Consider a horizontal barrier of length L, free on one end and pivoted at the other end, such that it can swivel horizontally around a vertical pivot. The hinge is constructed in such a way that the barrier is fixed perpendicularly to the upper end of a vertical hollow cylinder of diameter B and length H. This upper end is closed, the other end is open. With this open end the cylinder is placed over a vertical axis which is firmly anchored in the ground. Of course, the length of the axis is more than H and the diameter of the axis, b, is less than B.

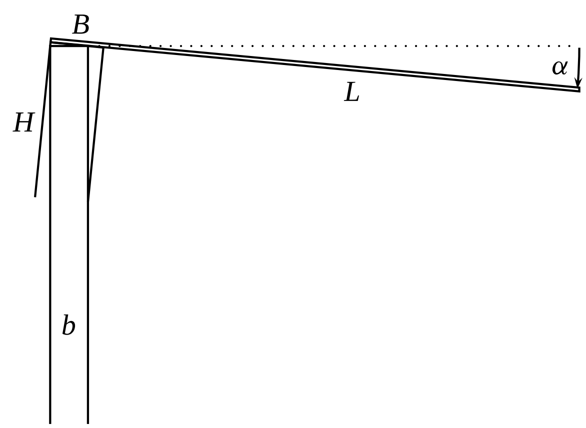


Figure 3.3: Slightly tilted barrier

Depending on the clearance between cylinder and axis, and the length of the cylinder, the free end of the barrier (which is otherwise perfectly stiff) will lean down from the exactly horizontal position. The question is: how much will this be.

You may assume that the construction is reasonable. In other words, the clearance will be small but not very small, and the length of the cylinder is ample.

Try to solve the problem geometrically exactly. It is possible to reformulate the problem as one of finding a zero of a 4-th order polynomial equation in $\sin\alpha$, where α is the angle of barrier with horizon. Conclude that the solution is difficult and clumsy.

Then try to make reasonable approximations and construct an adequate and transparent approximate solution.

3.4.5.6

Find an asymptotic approximation, for $\varepsilon \to 0$, of each solution $y = y(x, \varepsilon)$ of

$$y^2 + (1 + \varepsilon + x)y + x = 0, \quad \text{for } 0 < x < 1,$$

and determine if it is uniform in x over the indicated interval.

3.4.5.7 The Lagrange points of the reduced three body problem

Consider the so-called *Restricted Three Body Problem* consisting of a very small object moving in the gravity field of a system of two bodies, moving in a circle around their center of gravity. This gravity field (in a co-rotating coordinate system) has 5 points, where the sum of gravities and centrifugal forces cancel each other. Here, the small object could remain stationary (motionless in the co-rotating coordinates). These points are called the *Lagrange points* or *libration points*. Two of them can be given analytically exactly. If the mass ratio of both bodies is small, the location of the other three Lagrange points can be given asymptotically.

Consider the three masses M_1 (big), M_2 (small) and M_3 (negligible). The two masses M_1 and M_2 are assumed to be in circular orbits around their center of mass. The third mass M_3 is so small that it does not influence the motion of M_1 and M_2. We make dimensionless such that $M_1 = 1$ (the Sun, say) and $M_2 = \mu$ (Earth or Jupiter, say). μ is small but not negligible ($3.03591 \cdot 10^{-6}$ for the Earth-Sun system). $M_3 = 0$ (satellite, small planet) is negligibly small. Gravitational constant becomes $G = 1$ and the orbital period is 2π. The radii of the orbits around the center of gravity of M_1 is μ, and of M_2 is $1 - \mu$.

Introduce a coordinate system with the origin in the center of gravity, and co-rotates with M_1 and M_2. In this system, M_1 has fixed coordinates $(-\mu, 0)$ and M_2 has $(1 - \mu, 0)$. The equations of motion of M_3 in (x, y, z) are now

$$\ddot{x} - 2\,\dot{y} = \frac{\partial\Omega}{\partial x}, \qquad \ddot{y} + 2\,\dot{x} = \frac{\partial\Omega}{\partial y} \qquad \ddot{z} = \frac{\partial\Omega}{\partial z}$$

where

$$\begin{aligned}
\Omega &= \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{R_1} + \frac{\mu}{R_2} + \tfrac{1}{2}\mu(1-\mu),\\
R_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2},\\
R_2 &= \sqrt{(x-1+\mu)^2 + y^2 + z^2}.
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\partial\Omega}{\partial x} &= x - (1-\mu)\frac{x+\mu}{R_1^3} - \mu\frac{x-1+\mu}{R_2^3},\\
\frac{\partial\Omega}{\partial y} &= y - (1-\mu)\frac{y}{R_1^3} - \mu\frac{y}{R_2^3},\\
\frac{\partial\Omega}{\partial z} &= -(1-\mu)\frac{z}{R_1^3} - \mu\frac{z}{R_2^3}.
\end{aligned}$$

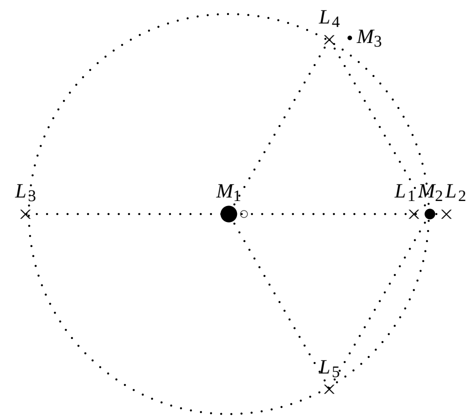


Figure 3.4: Sketch of restricted three-body problem with 5 Lagrange points and origin.

The 5 stationary points of this system are called “Lagrange points” or “libration points”. They are given by the system $\nabla\Omega = 0$, or

$$\begin{aligned}
x\left(1-\frac{1-\mu}{R_1^3}-\frac{\mu}{R_2^3}\right)-(1-\mu)\mu\left(\frac{1}{R_1^3}-\frac{1}{R_2^3}\right) &= 0,\\
y\left(1-\frac{1-\mu}{R_1^3}-\frac{\mu}{R_2^3}\right) &= 0,\\
z\left(\quad-\frac{1-\mu}{R_1^3}-\frac{\mu}{R_2^3}\right) &= 0.
\end{aligned}$$

All solutions are found in the plane $z = 0$, since $(1-\mu)R_1^{-3}+\mu R_2^{-3} > 0$. Lagrange points L_1, L_2, and L_3 are located on the line $y = 0$ (see below), but these are not the only solutions. The second factor of the y-equation may also vanish, in which case the x-equation simplifies to the condition $R_1 = R_2 = 1$. This then gives rise to the points L_4 and L_5, which are explicitly given by

$$x_{4,5} = \tfrac{1}{2}-\mu, \quad y_{4,5} = \pm\tfrac{1}{2}\sqrt{3}.$$

The other three points, L_1, L_2 and L_3, are located on the line $y = 0$, *i.e.* given by $y_1 = y_2 = y_3 = 0$ (the colinear libration points). The resulting x-equation can not be simplified further, but may be solved asymptotically for small μ. We have

$$x-(1-\mu)\frac{x+\mu}{|x+\mu|^3}-\mu\frac{x-1+\mu}{|x-1+\mu|^3} = 0.$$

Verify that the three coordinates x_3, x_4, and x_5 are given asymptotically by

$$\begin{aligned}
x_1 &= \quad 1-(\tfrac{1}{3}\mu)^{1/3}+\tfrac{1}{3}(\tfrac{1}{3}\mu)^{2/3}+O(\mu)\\
x_2 &= \quad 1+(\tfrac{1}{3}\mu)^{1/3}+\tfrac{1}{3}(\tfrac{1}{3}\mu)^{2/3}+O(\mu)\\
x_3 &= -1-\tfrac{5}{12}\mu+\tfrac{1127}{20736}\mu^3+O(\mu^4)
\end{aligned}$$

3.4.6 Solving differential equations asymptotically

3.4.6.1

Find a two-term asymptotic expansion, for $\varepsilon \to 0$, of the solution $y = y(x, \varepsilon)$ of the following problems.

(a) $y'' + \varepsilon y' - y = 1$, where $y(0) = y(1) = 1$.

(b) $y'' + y + y^3 = 0$, where $y(0) = 0$ and $y(\frac{1}{2}\pi) = \varepsilon$.

3.4.6.2 A car changing lanes

A car rides along a double lane straight road given by $-\infty < x < \infty$, $-2b \leqslant y \leqslant 2b$. The position of the car at time t is given by

$$x = \xi(t), \quad y = \eta(t).$$

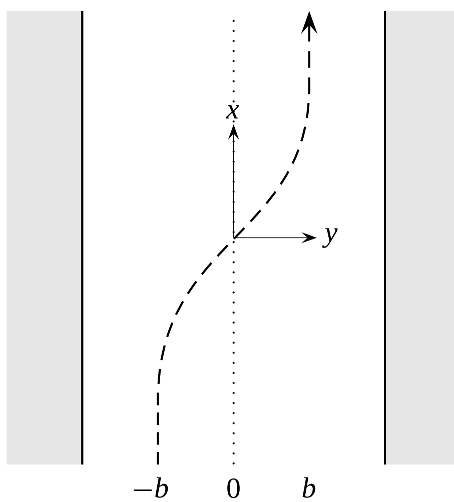


Figure 3.5: The trajectory of a car that changes lane

For $x \to -\infty$, the car is at $y = -b$, but near $x = 0$ it changes lane and shifts smoothly to $y = b$ according to a trajectory given by

$$\eta(t) = F(\xi(t)),$$

where F is given and $\xi = \xi(t)$ is to be found under the condition that all along the trajectory, the car travels with the same speed V, so

$$\dot{\xi}(t)^2 + \dot{\eta}(t)^2 = V^2, \quad \text{and so} \quad \dot{\xi}(t)^2 + F'(\xi)^2 \dot{\xi}(t)^2 = V^2.$$

Note that both F and its argument x have dimension "length", so if F describes a changes of the order of b over a distance of the order of L, we should be able to write F as

$$F(x) = bf(x/L)$$

for $f = O(1)$. Take for definiteness $\xi(0) = 0$, and

$$f(z) = \tanh(z) \quad \text{where} \quad f'(z) = 1 - f(z)^2.$$

We assume that the change of lane happens gradually, such that

$$\varepsilon = \frac{b}{L} \ll 1.$$

a. Make the problem dimensionless by the inherent length scale b and corresponding time scale b/V. Write $\xi = bX$. Note the appearance of the small parameter ε. Do you see the appearance of a term of the form $f'(\varepsilon X)$? If we expand this for small ε we obtain something like

$$f'(\varepsilon X) = f'(0) + \varepsilon X f''(0) + \dots$$

which is already incorrect for $X = O(1/\varepsilon)$, the order of magnitude we are interested in! Therefore this choice is NOT clever. Indeed, b is not the typical length scale for ξ.

b. Make the problem dimensionless by the inherent length scale L and corresponding time scale L/V. Write $\xi = LX$ and $t = (L/V)\tau$.

c. By separation of variables we can write τ as a function, in the form of an integral, of X. Otherwise, it is impossible to find an explicit expression for X. Therefore, we will try to find an asymptotic expansion for small ε by assuming the Poincaré expansion

$$X(\tau, \varepsilon) = X_0(\tau) + \varepsilon^2 X_1(\tau) + O(\varepsilon^2),$$

and substitute this in the equation, and expand the equation also asymptotically. Find the first two terms. Do you see why we can expand in powers of ε^2 rather than (for example) ε?

Hint: note that for small δ we approximate $(1+\delta)^a = 1 + a\delta + O(\delta^2)$, and

$$\int \tanh(x)^{2n}\,\mathrm{d}x = x - \tanh(x) - \tfrac{1}{3}\tanh(x)^3 - \tfrac{1}{5}\tanh(x)^5 - \cdots - \tfrac{1}{2n-1}\tanh(x)^{2n-1}$$

such that

$$\int (1-\tanh(x)^2)^1\,\mathrm{d}x = \tanh(x)$$
$$\int (1-\tanh(x)^2)^2\,\mathrm{d}x = \tanh(x) - \tfrac{1}{3}\tanh(x)^3$$
$$\int (1-\tanh(x)^2)^3\,\mathrm{d}x = \tanh(x) - \tfrac{2}{3}\tanh(x)^3 + \tfrac{1}{5}\tanh(x)^5$$

3.4.7 A water-bubbles mixture

A mixture of water and air (in the form of bubbles) with volume fraction α air and volume fraction $1-\alpha$ water, has a mean density ρ and sound speed c given by

$$\rho = \alpha\rho_a + (1-\alpha)\rho_w, \qquad \frac{1}{\rho c^2} = \frac{\alpha}{\rho_a c_a^2} + \frac{1-\alpha}{\rho_w c_w^2}.$$

Typical values are $\rho_w = 1000$ kg/m^3, $\rho_a = 1.2$ kg/m^3, $c_w = 1470$ m/s, $c_a = 340$ m/s. Develop strategies to approximate c for values of α, based on an inherent small problem parameter. When is c minimal? What is the effect of even a very small fraction of air (common in the wake of a ship's propeller, or in a fresh central heating system)?

3.4.8 A chemical reaction-diffusion problem (regular limit)

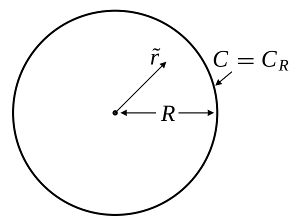
A catalytic reaction is a chemical reaction between reactants, of which one – the catalyst – returns after the reaction to its original state. Its rôle is entirely to enable the reaction to happen. An example of a catalyst is platinum. The primary reactant is usually a liquid or a gas. As the catalyst and the reactant are immiscible, the reaction occurs at the catalyst surface, which is therefore made as large as possible. A way to achieve this is by applying the catalyst in the pores of porous pellets in a so-called fixed bed catalytic reactor. The reactant diffuses from the surface to the inside of the pellet. Meanwhile, being in contact with the catalyst, the reactant is converted to the final product.

Assume reactant A reacts to product B at the pellet pores surface via an nth-order, irreversible reaction

$$A \stackrel{k}{\to} B$$

with concentration inside the pellet $C = [A]$ mol/m^3, production rate kC^n mol/m^{3}s and rate constant k. This reaction acts as a sink term for A. Under the additional assumption of a well stirred fluid in order to maintain a constant concentration $C = C_R$ at the outer surface of spherically shaped pellets, we obtain the following unsteady reaction-diffusion equation:

$$\begin{aligned} &\frac{\partial C}{\partial t} - \nabla\boldsymbol{\cdot}(D\nabla C) = -kC^n, \quad 0 < \tilde{r} < R, t > 0, \\ &C(r,0) = 0, \quad 0 < \tilde{r} < R, \\ &C(R,t) = C_R,\ \frac{\partial}{\partial \tilde{r}}C(0,t) = 0, \quad t > 0, \end{aligned}$$



where D is the diffusion coefficient of C inside the pellet. After sufficiently long time the concentration C attains a steady state distribution within the pellet. Assuming spherical symmetry and a constant diffusion coefficient D, we have the stationary reaction-diffusion equation

$$\begin{aligned} &D\frac{1}{\tilde{r}^2}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\Big(\tilde{r}^2\frac{\mathrm{d}C}{\mathrm{d}\tilde{r}}\Big) = kC^n, \quad 0 < \tilde{r} < R, \\ &C(R) = C_R,\ \frac{\mathrm{d}}{\mathrm{d}\tilde{r}}C(0) = 0. \end{aligned}$$

The net mass flux into the pellet, an important final result, is given by $4\pi R^2 D\frac{\mathrm{d}}{\mathrm{d}r}C(R)$ (Fick's law).

We make the problem dimensionless as follows:

$$c = \frac{C}{C_R}, \quad r = \frac{\tilde{r}}{R}, \quad \phi^2 = \frac{kR^2C_R^{n-1}}{D},$$

such that

$$\begin{aligned} &\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\frac{\mathrm{d}c}{\mathrm{d}r}\Big) = \phi^2 c^n, \quad 0 < r < 1, \\ &c(1) = 1, \ \ c'(0) = 0, \end{aligned}$$

where the prime ($'$) denotes differentiation with respect to r, ϕ is called the Thiele modulus, and reaction order $n = 1, 2, 3, \ldots$.

We are interested in the asymptotic behaviour of c for $\varepsilon = \phi^2 \to 0$. Assume a regular Poincaré expansion of c in powers of ε and find the first three terms. *Hint.* Introduce $y = rc$.

Chapter 4

Method of Slow Variation

4.1 Theory

The name *Method of Slow Variation* was coined relatively recently in 1987 by Milton Van Dyke [6].

4.1.1 General procedure

Suppose we have a function $\varphi(\boldsymbol{x}, \varepsilon)$ of spatial coordinates $\boldsymbol{x} \in \mathcal{V}$ and a small parameter ε, such that the typical variation in one direction, say x, is of the order of length scale ε^{-1}. For example,

$$\varphi(x, \varepsilon) = 1 + \varepsilon \sin(\varepsilon x + \varepsilon), \qquad \text{or} \qquad \varphi(x, \varepsilon) = \frac{x}{\varepsilon^{-2} + x^2}$$

along $\mathcal{V} = \mathbb{R}$. Roughly speaking, this amounts to something like $\frac{\partial}{\partial x}\varphi = O(\varepsilon\varphi)$. However, if φ is zero, or much smaller or larger than the varying part of φ, this is not what we mean. Rather than formulating a mathematically precise but cumbersome definition, we will express this behaviour most conveniently by writing

$$\varphi(x, y, z, \varepsilon) = \Phi(\varepsilon x, y, z, \varepsilon)$$

under the assumption that

$$\Phi(X, y, z, \varepsilon) = O(\mu_0(\varepsilon))$$

uniformly in its domain of definition. Now if we were to expand $\Phi(\varepsilon x, y, z, \varepsilon)$ for small ε, we might, for example by some Taylor-like expansion in ε, get something like (assume $\mu_0 = 1$)

$$\Phi(\varepsilon x, y, z, \varepsilon) = \Phi(0, y, z; 0) + \varepsilon\big(x\Phi_x(0, y, z; 0) + \Phi_\varepsilon(0, y, z; 0)\big) + \ldots$$

which is only uniform in x on an interval $[0, L]$ if $L = O(1)$, while the inherent slow variation on the longer scale of $x = O(\varepsilon^{-1})$ would be masked. It is clearly much better to absorb the ε-dependence in εx into a new variable, and introduce the scaled variable $X = \varepsilon x$. The (assumed) regular expansion of $\Phi(X, y, z, \varepsilon)$

$$\Phi(X, y, z, \varepsilon) = \mu_0(\varepsilon)\varphi_0(X, y, z) + \ldots \tag{4.1}$$

now retains the slow variation in X in the shape functions of the expansion and remains valid for all X. In other words, the scaled variable X in combination with order function μ_0 yields with $\lim_{\varepsilon\to 0} \mu_0^{-1}\Phi$ the *distinguished limit* or *significant degeneration* of φ.

This situation frequently happens when the geometry involved is slender. The theory of one dimensional gas dynamics, lubrication flow, or sound propagation in horns (Webster's horn equation) are important examples, although they are usually derived not systematically, without explicit reference to the slender geometry. We will illustrate the method both for heat flow in a varying bar, quasi 1-D gas flow and the shallow water problem. A more advanced example, presented for illustration in section 2.1.5, is the weakly nonlinear theory for long water waves, resulting in the celebrated Korteweg-de × Vries equation.

4.1.2 Example: heat flow in a bar

Consider the stationary problem of the temperature distribution T in a long heat-conducting bar, with constant heat conductivity κ, outward surface normal $\boldsymbol{n}$, and slowly varying cross section $\mathcal{A}$. The bar is kept at a temperature difference such that a given heat flux is maintained, but is otherwise isolated. As there is no leakage of heat, the axial flux $\mathcal{F}$ along a cross section is constant. With spatial coordinates made dimensionless on a typical bar cross section D and the temperature on a typical temperature Θ_0, we can write the flux in a dimensionless form like $\mathcal{F} = \kappa\Theta_0 DQ$, and have the following equations and boundary conditions

$$\nabla^2 T = 0, \qquad \big[\nabla T \cdot \boldsymbol{n}\big]_{\text{surface}} = 0, \qquad \iint_{\mathcal{A}} \Big(-\frac{\partial T}{\partial x}\Big)\mathrm{d}S = Q.$$

(Note that only the derivatives of T play a role, so we may subtract any reference level and assume, for example, that T is at one of the ends equal to zero.) After integrating $\nabla^2 T$ over a slice $x_1 \leqslant x \leqslant x_2$, and applying Gauss's theorem, we find that the axial flux Q is indeed independent of x.

We will assume here the cross section and the temperature field circular symmetric, but that is not a necessary simplification for a manageable analysis. As a result we have in cylindrical coordinates (x, r, θ)

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0, \qquad \frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial r}n_r = 0, \qquad 2\pi \int_0^R r\frac{\partial T}{\partial x}\,\mathrm{d}r = -Q.$$

The typical length scale L of diameter variation is assumed to be much larger than a diameter D. We introduce their ratio as the small parameter $\varepsilon = D/L$, and write for the bar surface

$$S(X, r) = r - R(X) = 0, \quad X = \varepsilon x,$$

where (x, r, θ) form a cylindrical coordinate system (see Figure 4.1). By writing R as a continuous function of slow variable X, rather than x, we have made our formal assumption of slow variation explicit in a convenient and simple way, since $R_x = \varepsilon R_X = O(\varepsilon)$. From calculus we know, that ∇S is a normal of the surface $S = 0$. So we can write

$$\boldsymbol{n} \sim \nabla S, \quad \text{or} \quad n_x\boldsymbol{e}_x + n_r\boldsymbol{e}_r \sim -\varepsilon R_X\boldsymbol{e}_x + \boldsymbol{e}_r.$$

The crucial step will now be the assumption that the temperature is *only* affected by the geometric variation induced by R. Any initial or entrance effects are ignored or have disappeared. As a result, in the limit of small ε,

the temperature field $T(x, r, \varepsilon) = \tilde{T}(X, r, \varepsilon)$ has a regular expansion[1] in variable X,

[1] *i.e.* on the whole bar

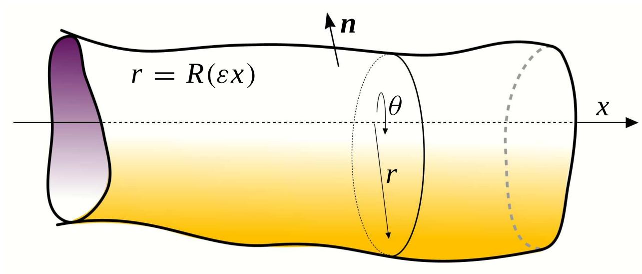


Figure 4.1: Slowly varying bar.

rather than x – in other words: $\tilde{T}$ yields the distinguished limit of T – and

$$\text{its axial gradient scales on } \varepsilon, \text{ as } \frac{\partial T}{\partial x} = \varepsilon \frac{\partial \tilde{T}}{\partial X} = O(\varepsilon).$$

For simplicity we will write in the following T, instead of $\tilde{T}$. If we rewrite the equations from x into X, we obtain the rescaled heat equation

$$\varepsilon^2 T_{XX} + \frac{1}{r}\big(rT_r\big)_r = 0. \tag{$*$}$$

At the wall $r = R(X)$ the boundary condition of vanishing heat flux is

$$-\varepsilon^2 T_X R_X + T_r = 0. \tag{$\dagger$}$$

The flux condition, for lucidity rewritten with $Q = 2\pi\varepsilon q$, is given by

$$\int_0^{R(X)} r\frac{\partial T}{\partial X}\,\mathrm{d}r = -q.$$

This problem is too difficult in general, so we try to utilize the small parameter ε in a systematic manner. From the flux condition, it seems that $T = O(1)$. Since the perturbation terms are $O(\varepsilon^2)$, we assume the asymptotic expansion of Poincaré-type, with shape functions of (X,r), *not* of (x,r)

$$T(X,r,\varepsilon) = T_0(X,r) + \varepsilon^2 T_1(X,r) + O(\varepsilon^4).$$

(Note: this is essentially a modelling assumption and not necessarily possible for any problems.) After substitution in equation ($*$) and boundary condition (†), further expansion in powers of ε^2 and equating like powers of ε, we obtain to leading order the following equation in r

$$(rT_{0,r})_r = 0 \qquad \text{with} \quad T_{0,r} = 0 \quad \text{at} \;\; r = R(X) \;\; \text{and regular at} \;\; r = 0.$$

An obvious solution is $T_0(X,r)$ is constant. Since X is present as parameter we have thus

$$T_0 = T_0(X).$$

We can substitute this directly in the flux condition, to find

$$\frac{1}{2}R^2(X)\frac{\mathrm{d}T_0}{\mathrm{d}X} = -q$$

and therefore

$$T_0(X) = T_{\mathrm{in}} - \int_0^X \frac{q}{\frac{1}{2}R^2(\xi)}\,\mathrm{d}\xi.$$

We can go on to find the next term T_1, but this leading order solution contains already most of the physical information.

In summary: we assumed that the slowly varying bar induces a slowly varying temperature distribution. This is not always true, but depends on the type of physical phenomenon. Then we rescaled the equations such that we used this slow variation. After assuming an asymptotic expansion of the solution we obtained a simplified sequence of problems. The original partial differential equations simplified to ordinary differential equations, which are far easier to solve.

It should be noted that we did not include in our analysis any boundary conditions at the ends of the bar. It is true that the present method fails here. The found solution is uniformly valid on $\mathbb{R}$ (since $R(X)$ is assumed continuous and independent of ε), but only as long as we stay away from any end. Near the ends the boundary conditions induce x-gradients of $O(1)$ which makes the prevailing length scale again x, rather than X. This region is asymptotically of boundary layer type, and should be treated differently (see below).

4.2 Method of Slow Variation: Assignments

4.2.1 Heat flux in a bar

Consider the stationary two-dimensional problem of a long heat-conducting bar, slowly varying in diameter, which is kept at both ends at a different temperature, and which is otherwise thermally isolated. We will not consider the neighbourhood of the ends, and therefore we will not explicitly apply boundary conditions at the ends. Instead, we will assume a given axial heat flux.

In *dimensional* form, we have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{along} \quad 0 < x < \ell,\ 0 \leqslant y \leqslant Hh\big(\tfrac{x}{L}\big)$$

with $\varepsilon = H/L$ is small, ℓ is large enough, $h = O(1)$ is a smooth, strictly positive function, and a flux

$$-\int_0^{Hh} \kappa \frac{\partial T}{\partial x}\,\mathrm{d}y = Q$$

is prescribed. Make the problem dimensionless on H and a suitable temperature. Write the boundary condition of thermal isolation (flux $\sim \nabla T \cdot \vec{\boldsymbol{n}} = 0$) in terms of h.

Apparently, the essential co-ordinate in x-direction is x/L, and significant changes in x-direction are felt only on a length scale $x = O(L^{-1})$, so we introduce a slow axial variable.

Assume that the field varies axially in this variable (ℓ is large enough so any end-effects are local and assumed irrelevant here).

Solve the problem to leading order of an assumed asymptotic expansion of T in powers of ε.

4.2.2 Lubrication flow

Lubrication theory deals with a viscous flow (not-large Reynolds number) through a narrow channel of slowly varying cross section.

Consider steady flow in a two-dimensional narrow channel, with prescribed volume flux. In practice this flux is created by a prescribed pressure difference or pressure gradient, but by using the flux here, we can estimate the typical flow velocity and thus the Reynolds number.

If we make dimensionless on the channel height, and scale the pressure gradient such that viscous forces are balanced by the (externally applied) pressure gradient, we obtain in dimensionless form (check!)

$$Re\Big(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) + \frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad Re\Big(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\Big) + \frac{\partial p}{\partial y} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

for the velocity (u, v) and pressure p in the channel

$$-\infty < x < \infty, \qquad g(\varepsilon x) \leqslant y \leqslant h(\varepsilon x).$$

(End conditions in x are not important.) Boundary conditions are no slip at the walls:

$$u = v = 0 \quad \text{at } y = g(\varepsilon x),\ \text{and } y = h(\varepsilon x)$$

such that $0 < h - g = O_s(1)$. Furthermore, the flow is assumed to consitute the flux

$$\int_{g(\varepsilon x)}^{h(\varepsilon x)} u(x, y)\,\mathrm{d}y = 1.$$

(If required, we can fix the pressure somewhere, for example $p(x = 0) = 0$, but this is not important.)

Apparently, the essential co-ordinate in x-direction is εx, and significant changes in x-direction are felt only on a length scale $x = O(\varepsilon^{-1})$, so we rewrite $X = \varepsilon x$.

Assume that the field varies slowly in X (any end-effects are local and irrelevant for the x's considered).

How do we scale u, v, p? Do not forget the fact that $Re \leqslant O(1)$, and that a pressure gradient is necessary to have a flow, while further the crosswise velocity v will be much smaller than the axial velocity u.

Assume for scaled u, v, p an obvious asymptotic expansion in ε, and solve up to leading order.

4.2.3 Quasi 1D gas dynamics

Consider a compressible, subsonic inviscid irrotational steady flow through a slowly varying cylindrical duct, given dimensionless by $r = R(\varepsilon x)$. The flow is assumed nearly uniform. Because of symmetry, it is assumed to be independent of the circumferential co-ordinate θ. As the flow is irrotational, we can assume a potential ϕ for the velocity $\boldsymbol{v}$. Density ρ and pressure p are related via thermodynamic relations of isentropy.

This type of flow is called: 1D gas dynamics. A better name would be: quasi-1D gas dynamics.

In dimensionless form, the flow is described by the following equations. Inside the duct $0 \leqslant r \leqslant R(\varepsilon x) = O(1)$ we have the mass equation, Bernoulli's equation and the isentropic relation

$$\nabla\cdot(\rho\boldsymbol{v}) = \frac{\partial}{\partial x}(\rho u) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v) = 0, \quad \text{where} \quad \boldsymbol{v} = \nabla\phi = \left(\tfrac{\partial}{\partial x}\phi\right)\boldsymbol{e}_x + \left(\tfrac{\partial}{\partial r}\phi\right)\boldsymbol{e}_r = u\boldsymbol{e}_x + v\boldsymbol{e}_r$$

$$\tfrac{1}{2}|\boldsymbol{v}|^2 + \frac{c^2}{\gamma - 1} = E, \quad \text{a constant} \quad O(1)$$

$$c^2 = \gamma\frac{p}{\rho} = \rho^{\gamma-1} = O(1)$$

The duct walls, with normal vector $\boldsymbol{n}$, are impermeable, so

$$\boldsymbol{v}\cdot\boldsymbol{n} = \nabla\phi\cdot\boldsymbol{n} = 0 \quad \text{at} \quad r = R(\varepsilon x),$$

while a mass flux F is given by

$$2\pi\int_0^{R(\varepsilon x)} \rho(x, r)u(x, r)r\,\mathrm{d}r = F = O(1).$$

The thermodynamical properties are fixed by the Bernoulli constant E. The physical parameter γ, which is just a constant, is typically for air equal to 1.4. The auxiliary variable c denotes the sound speed, and is otherwise unimportant.

Apparently, the essential co-ordinate in x-direction is εx, and significant changes in x-direction are felt only on a length scale $x = O(\varepsilon^{-1})$, so we rewrite $X = \varepsilon x$.

How do we scale ϕ, u, v, ρ, p, c? Pay particularly attention to the fact that from the flux condition it follows that $u = O(1)$, while also $u = \frac{\partial}{\partial x}\phi$. Do not forget the fact that a pressure gradient is necessary to have a flow, while further the crosswise velocity v will be much smaller than the axial velocity u.

Assume that the field varies slowly in X (any end-effects are local and irrelevant for the x's considered).

Assume for scaled ϕ, u, v, ρ, p, c an obvious asymptotic expansion in ε, and determine the prevailing equations to leading order. Solve the equations for the velocity. For density ρ we are left with an algebraic equation that cannot be solved explicitly.

4.2.4 Webster's horn

Consider acoustic waves of fixed frequency ω through a slowly varying horn (duct). The typical wave length λ is long, *i.e.* of the same order of magnitude as the typical length scale L of the duct diameter variations. For simplicity we consider a two-dimensional horn, with a constant lower wall given by $\tilde{y} = 0$ and an upper wall given by $\tilde{y} = Hh(\tilde{x}/L)$, where $h = O(1)$ is dimensionless and $H \ll L$.

The sound field is given by the velocity potential $\tilde{\phi}$, where velocity is $\tilde{\boldsymbol{v}} = \tilde{\nabla}\tilde{\phi}$ (and pressure $\tilde{p} = -\mathrm{i}\omega\rho_0\tilde{\phi}$ but this is here unimportant), obeying the reduced wave equation (Helmholtz equation)

$$\tilde{\nabla}^2\tilde{\phi} + \tilde{k}^2\tilde{\phi} = 0, \quad \text{in } -\infty < \tilde{x} < \infty, \;\; 0 \leqslant \tilde{y} \leqslant Hh(\tilde{x}/L),$$

where $\tilde{k} = \omega/c$ is the free field wave number, which is equal to $2\pi/\lambda$.

The wall (with normal vectors $\boldsymbol{e}_y$ and $\boldsymbol{n}$) are impermeable, so we have the boundary conditions

$$\tilde{\boldsymbol{v}} \cdot \boldsymbol{e}_y = 0 \quad \text{at } \tilde{y} = 0, \qquad \tilde{\boldsymbol{v}} \cdot \boldsymbol{n} = 0 \quad \text{at } \tilde{y} = Hh(\tilde{x}/L).$$

Assume that there is a sound field (the problem is linear, so it's enough to assume that $\tilde{\phi} \not\equiv 0$).

Make the lengths in the problem dimensionless on the typical duct height H and $\tilde{\phi}$ on an (unimportant) reference value Φ. Verify that the equations remain the same. Introduce the small parameter $\varepsilon = H/L$.

Apparently, the $\tilde{x}$ variations scale on L, and so the essential co-ordinate in x-direction is εx. Significant changes in x-direction are felt only on a length scale $x = O(\varepsilon^{-1})$, and so we rewrite $X = \varepsilon x$.

Note that the dimensionless $k = O(\varepsilon)$, so we scale $k = \varepsilon\kappa$.

Assume that the field varies slowly in X (any end-effects are local and irrelevant here).

Assume in scaled coordinates for ϕ an obvious asymptotic expansion in ε, and derive the equation for (leading order) ϕ_0. This equation is called "Webster's equation".

Solve this equation for $h(z) = \mathrm{e}^{2\alpha z}$.

4.2.5 Shallow water waves along a varying bottom

Consider the following inviscid incompressible irrotational 2D steady water flow in (x, z)-domain along a slowly varying bottom. The bottom is given by $z = b(x/L)$, where L is a typical length scale along which bottom variations occur. The water level is given by $z = h(x)$.

The velocity vector $\boldsymbol{v}$ can be given by a potential ϕ

$$\boldsymbol{v} = \nabla\phi.$$

Conservation of mass requires

$$\nabla^2\phi = 0 \ \text{ for } \ -\infty < x < \infty, \ \ b < z < h.$$

Because of the assumptions we can integrate the momentum equation to Bernoulli's equation and obtain for pressure p

$$\tfrac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho_0} + gz = C,$$

where ρ_0 denotes the water density, g the acceleration of gravity, and C is a constant, related to the chosen reference pressure level.

At the impermeable bottom we have a vanishing normal component of the velocity, yielding the boundary condition

$$\nabla\phi \cdot \nabla(b - z) = \frac{\partial\phi}{\partial x}\frac{\partial b}{\partial x} - \frac{\partial\phi}{\partial z} = 0 \ \text{ at } \ z = b.$$

Since the water surface $z = h$ is a streamline, it follows that for a particle moving along $(x(t), z(t))$ with $z(t) = h(x(t))$ we have $\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$, leading to

$$\frac{\partial\phi}{\partial z} = \frac{\partial h}{\partial x}\frac{\partial\phi}{\partial x} \ \text{ at } \ z = h.$$

Furthermore, the water surface takes the pressure of the air above the water, say $p = p_a$, so

$$\tfrac{1}{2}|\nabla\phi|^2 + gh + \frac{p_a}{\rho_0} = C \ \text{ at } \ z = h.$$

The water flow is defined by a prescribed volume flux F, which is the same for all positions x.

$$\int_b^h \frac{\partial\phi}{\partial x}\,\mathrm{d}z = F.$$

By assuming far upstream a constant bottom level $b = b_\infty$, a constant water level $h = h_\infty = b_\infty + D_\infty$ and a uniform flow with velocity $U_\infty = F/D_\infty$, we can determine the Bernoulli constant in physical terms

$$C = \frac{p_a}{\rho_0} + \tfrac{1}{2}U_\infty^2 + gh_\infty.$$

Introduce $\varepsilon = D_\infty/L$ where ε is small.

a) Make the problem dimensionless. Scale lengths on D_∞, velocities on U_∞. Assume that the inverse-squared Froude number (or Richardson number) $\gamma = gD_\infty/U_\infty^2 = O(1)$.

b) Solve the problem to leading order for small ε by application of the Method of Slow Variation. Note that both ϕ and h are unknowns, and have to be expanded in ε. Bottom variation b and constants F and $C - p_a/\rho_0$, on the other hand, are given.

   Note. The very last equation cannot be integrated explicitly.

4.2.6 A laterally heated bar

A 2-dimensional slowly varying heat conducting bar is described by

$$-\infty < \tilde{x} < \infty, \quad \tilde{y}_0 + Hg(\tilde{x}/L) \leqslant \tilde{y} \leqslant \tilde{y}_0 + Hh(\tilde{x}/L),$$

where the geometries g and h are smooth functions of their argument. The bar is kept along the lower side at fixed temperature $\tilde{T}(\tilde{x}, \tilde{y}_0 + Hg) = \theta_0$, and along the upper side at fixed temperature $\tilde{T}(\tilde{x}, \tilde{y}_0 + Hh) = \theta_1$. This constitutes a stationary temperature distribution $\tilde{T}(\tilde{x}, \tilde{y})$, which satisfies the heat equation

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{y}^2} = 0.$$

a) Make the problem dimensionless. Scale lengths on H by $\tilde{x} = Hx$ and $\tilde{y} = \tilde{y}_0 + Hy$, and temperature by $\tilde{T} = \theta_0 + (\theta_1 - \theta_0)T$. Introduce the geometric ratio $\varepsilon = H/L$. Assume that ε is small. As the notation suggests, $g(z)$ and $h(z)$ do not depend on ε and $0 < h(z) - g(z) = O(1)$.

b) Assuming that T is slowly varying with geometry g and h in x (no end effects), solve the problem asymptotically for small ε to first and second order by application of the Method of Slow Variation.

Chapter 5

Method of Lindstedt-Poincaré

5.1 Theory

5.1.1 Secular behaviour with naive expansion

When we have a function y, depending on a small parameter ε, and periodic in t with fundamental frequency $\omega(\varepsilon)$, we can write y as a Fourier series

$$y(t,\varepsilon) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon)\, \mathrm{e}^{\mathrm{i}n\omega(\varepsilon)t} \tag{5.1}$$

If amplitudes and frequency have an asymptotic expansion for small ε, say

$$A_n(\varepsilon) = A_{n,0} + \varepsilon A_{n,1} + \ldots, \qquad \omega(\varepsilon) = \omega_0 + \varepsilon\omega_1 + \ldots, \tag{5.2}$$

we have a natural asymptotic series expansion for y of the form

$$y(t,\varepsilon) = \sum_{n=-\infty}^{\infty} A_{n,0}\, \mathrm{e}^{\mathrm{i}n\omega_0 t} + \varepsilon \sum_{n=-\infty}^{\infty} \Big(A_{n,1} + \mathrm{i}n\omega_1 t A_{n,0}\Big)\, \mathrm{e}^{\mathrm{i}n\omega_0 t} + \ldots \tag{5.3}$$

This expansion, however, is only uniform in t on an interval $[0,T]$, where $T = o(\varepsilon^{-1})$. On a larger interval, for example $[0,\varepsilon^{-1}]$, the asymptotic hierarchy in the expansion becomes invalid, because $\varepsilon t = O(1)$. This happens because of the occurrence of algebraically growing oscillatory terms, called "secular terms". *Secular* = occurring once in a century, and *saeculum* = generation, referring to their astronomical origin.

Definition. The terms proportional to $t^m \sin(n\omega_0 t)$, $t^m \cos(n\omega_0 t)$ are called "*secular terms*". More generally, the name refers to any algebraically growing terms that limit the region of validity of an asymptotic expansion.

5.1.2 General Procedure

So with a naive expansion of a periodic function we may expect secular behaviour that spoils regular behaviour for large values of t. Of course, we can accept our loss and limit the region of validity, but it is far better to apply first a coordinate transformation $\tau = \omega(\varepsilon)t$, introduce $Y(\tau,\varepsilon) = y(t,\varepsilon)$, and expand Y, rather than y, asymptotically. We get

$$Y(\tau,\varepsilon) = \sum_{n=-\infty}^{\infty} A_n(\varepsilon)\,\mathrm{e}^{in\tau} = \sum_{n=-\infty}^{\infty} A_{n,0}\,\mathrm{e}^{in\tau} + \varepsilon \sum_{n=-\infty}^{\infty} A_{n,1}\,\mathrm{e}^{in\tau} + \ldots \tag{5.4}$$

which is now, in variable τ, a *uniformly valid* approximation!

The method is called the *Lindstedt-Poincaré method* or the *method of strained coordinates*. In practical situations, the function $y(t,\varepsilon)$ is implicitly given, often by a differential equation, and to be found. A typical, but certainly not the only example [40] is a weakly nonlinear harmonic equation of the form

$$y'' + \varepsilon h(y, y') + \alpha^2 y = 0,$$

where h is assumed to allow the existence of one or more periodic solutions for $y = O(1)$ with frequency $\omega(\varepsilon) \approx \alpha$ for $\varepsilon \to 0$. In view of the above, it makes sense to construct an asymptotic approximation like $Y = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots$ with a rescaled variable $\tau = \omega t$. However, except for trivial situations, the frequency ω is unknown, and has to be found too. Therefore, when constructing the solution we have to allow for an unknown coordinate transformation. In order to construct the unknown $\omega(\varepsilon)$ we expand this in a similar way, for example like

$$\tau = (\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots)t \tag{5.5}$$

but details depend on the problem. Note that the only purpose of the scaling is to render the asymptotic expansion of Y regular, so it is no restriction to assume for ω_0 something convenient, like $\omega_0 = \alpha$. The other coefficients $\omega_1, \omega_2, \ldots$ are determined from the additional condition that the asymptotic hierarchy should be respected as long as possible. In other words, secular terms should not occur. We will illustrate this with the following classic example.

5.1.3 Example: the pendulum

Consider the motion of the pendulum, described[1] by the ordinary differential equation

$$\ddot{\theta} + K^2 \sin(\theta) = 0, \quad \text{with } \theta(0) = \varepsilon,\ \theta'(0) = 0,$$

where $0 < \varepsilon \ll 1$, $K = O(1)$. By elementary arguments (see section 9.2) it can be shown that periodic solutions exist. We note that $\theta = O(\varepsilon)$ so we scale $\theta = \varepsilon\psi$ to get (after dividing by ε)

$$\ddot{\psi} + K^2(\psi - \tfrac{1}{6}\varepsilon^2\psi^3 + \ldots) = 0, \quad \text{with } \psi(0) = 1,\ \psi'(0) = 0.$$

If we are interested in a solution only up to $O(\varepsilon^2)$ we can obviously ignore the higher order terms indicated by the dots, to get a version of the Duffing equation.

[1] The equation may be simpified by rescaling time by $\tilde{t} = Kt$, such that factor K^2 cancels out.

Following the above procedure, we introduce the transformation $\tau = \omega t$ to obtain

$$\omega^2 \psi'' + K^2\big(\psi - \tfrac{1}{6}\varepsilon^2\psi^3\big) = 0,$$

where the prime indicates now differentiation to τ. Since the essential small parameter is apparently ε^2, we expand

$$\omega = \omega_0 + \varepsilon^2\omega_1 + \ldots, \qquad \psi = \psi_0 + \varepsilon^2\psi_1 + \ldots,$$

and find, after substitution, the equations for the first two orders

$$\begin{aligned} &\omega_0^2\psi_0'' + K^2\psi_0 = 0, && \psi_0(0) = 1,\ \psi_0'(0) = 0,\\ &\omega_0^2\psi_1'' + K^2\psi_1 = -2\omega_0\omega_1\psi_0'' + \tfrac{1}{6}K^2\psi_0^3, && \psi_1(0) = 0,\ \psi_1'(0) = 0. \end{aligned}$$

Note that we are relatively free to choose ω_0, as long as it is $O(1)$. (It is only a coordinate transformation that would automatically be compensated in the equation.) Clearly, a good choice is $\omega_0 = K$ because this simplifies the formulas greatly. The solution ψ_0 is then

$$\psi_0 = \cos\tau, \quad \omega_0 = K,$$

leading to the following equation for ψ_1

$$\psi'' + \psi_1 = 2K^{-1}\omega_1\cos\tau + \tfrac{1}{6}\cos^3\tau = 2K^{-1}\omega_1\cos\tau + \tfrac{1}{8}\cos\tau + \tfrac{1}{24}\cos 3\tau,$$

using section (9.4) to expand $\cos^3\tau$. At this point it is essential to observe that the right-hand-side consists of two forcing terms: one with frequency 3 and one with 1, the resonance frequency of the left-hand-side. This resonance would lead to secular terms, as the solutions will behave like $\tau\sin(\tau)$ and $\tau\cos(\tau)$. This would spoil our approximation if we had no further degrees of freedom. However, this is where our rescaled time comes in! We know that by scaling with the *correct* frequency ω of the system there will be no secular terms. So we have to choose ω_1 such, that no secular terms arise.

Therefore, in order to suppress the occurrence of secular terms, the amplitude of the resonant forcing term should vanish, which yields the next order terms ω_1 and ψ_1. We thus have

$$\omega_1 = -\tfrac{1}{16}K$$

leading to

$$\psi_1 = A_1\cos\tau + B_1\sin\tau - \tfrac{1}{192}\cos 3\tau.$$

With the initial conditions this is

$$\psi_1 = \tfrac{1}{192}\big(\cos\tau - \cos 3\tau\big).$$

Altogether we have eventually

$$\theta(t) = \varepsilon\cos\tau + \tfrac{1}{192}\varepsilon^3\big(\cos\tau - \cos 3\tau\big) + O(\varepsilon^5), \quad \tau = K\big(1 - \tfrac{1}{16}\varepsilon^2 + O(\varepsilon^4)\big)t.$$

5.2 Method of Lindstedt-Poincaré: Assignments

5.2.1 A quadratically perturbed harmonic oscillator

Consider the following problem for $y(t,\varepsilon)$

$$y'' + y - y^2 = 0, \quad \text{with} \quad y(0) = \varepsilon, \quad y'(0) = 0$$

asymptotically for small positive parameter ε.

i) Show by phase plane considerations (section 9.1) that y is periodic for small ε.
ii) Determine a three term straightforward expansion and discuss its uniformity for large t.
iii) Construct by means of the Lindstedt-Poincaré method ("method of strained coordinates") a three term approximate solution.

5.2.2 A weakly nonlinear harmonic oscillator

Consider the following problem for $y(t,\varepsilon)$

$$y'' + (1 + y'^2)y = 0, \quad \text{with} \quad y(0) = \varepsilon, \quad y'(0) = 0$$

asymptotically for small positive parameter ε.

i) Determine a two term straightforward expansion and discuss its uniformity for large t.
ii) Construct by means of the Lindstedt-Poincaré method ("method of strained coordinates") a two term approximate solution.

5.2.3 A weakly nonlinear, quadratically perturbed harmonic oscillator

Consider the system governed by the equation of motion

$$y'' + y + \varepsilon\alpha y^2 = 0, \quad y(0) = 0, \quad y'(0) = \beta,$$

asymptotically for $\varepsilon \to 0$, where $\alpha = O(1)$. *Hint:* rescale $y := \beta y$ and $\alpha\beta\varepsilon := \varepsilon$.

i) Show by phase plane considerations (section 9.1) that y is periodic for small ε.
ii) Determine a three term straightforward expansion and discuss its uniformity for large t.
iii) Determine a three term expansion, valid for large t, by means of the Lindstedt-Poincaré method.

5.2.4 A coupled nonlinear oscillator

Determine a first-order uniformly valid expansion for the *periodic* solution of

$$\begin{aligned} u'' + u &= \varepsilon(1 - z)u' \\ cz' + z &= u^2 \end{aligned}$$

asymptotically for $\varepsilon \to 0$, where $c = O(1)$ is a positive constant and $u, z = O(1)$. You are free to make the solution unique in any convenient way, as long as it is periodic.

5.2.5 A weakly nonlinear 4th order oscillator

Determine a periodic solution to $O(\varepsilon)$ of the problem

$$u'''' + u'' + u' + u = \varepsilon(1 - u^2 - (u')^2 - (u'')^2)(u'' + u')$$

asymptotically for $\varepsilon \to 0$, where $u = O(1)$.

5.2.6 A weakly nonlinear oscillator

Use Lindstedt-Poincaré's method to get a two-term asymptotic approximation $y = y(t)$ to the problem

$$y'' + y = \varepsilon y y'^2, \quad y(0) = 1, \ \ y'(0) = 0.$$

5.2.7 The Van der Pol oscillator

Consider the weakly nonlinear oscillator, described by the Van der Pol equation, for variable $y = y(t, \varepsilon)$ in t:

$$y'' + y - \varepsilon(1 - y^2)y' = 0$$

asymptotically for small positive parameter ε.

Construct by means of the Lindstedt-Poincaré method ("method of strained coordinates") a second-order (three term) approximation of a *periodic* solution.

Note that not all solutions are periodic (see for example the phase portrait in figure 9.2), so you have to make sure to start on the right trajectory. Apart from this, you are free to make the solution unique in any convenient way. Take for example initial conditions

$$y(0) = A(\varepsilon), \ \ y'(0) = 0$$

with $A(\varepsilon)$ to be determined.

5.2.8 A variant of the Van der Pol oscillator

The same as above for

$$y'' + y - \varepsilon(1 - y^4)y' = 0$$

5.2.9 Another weakly nonlinear oscillator

For parameter $\beta = O(1)$

$$y'' + y + \varepsilon(y'^2 + \beta y^3) = 0$$

Chapter 6

Matched Asymptotic Expansions

6.1 Theory

6.1.1 Singular perturbation problems

If the solution of the problem considered does not allow a regular expansion, the problem is singular and the solution has no uniform Poincaré expansion in the same variable. We will consider two classes of problems. In the first one the singular behaviour is of boundary layer type and the solution can be built up from locally regular expansions. The solution method is called “method of matched asymptotic expansions”. In the other one more time or length scales occur together and a solution is constructed by considering these length scales as if they were independent. The solution method is called “method of multiple scales”.

6.1.2 Matched Asymptotic Expansions

Very often it happens that a simplifying limit applied to a more comprehensive model gives a correct approximation for the main part of the domain, but not everywhere: the limit is *non-uniform*. This non-uniformity may be in space, in time, or in any other variable. For the moment we think of non-uniformity in space, let’s say a small region near $x = 0$. If this region of non-uniformity is crucial for the problem, for example because it contains a boundary condition, or a source, the primary reduced problem (which does not include the region of non-uniformity) is not sufficient. This, however, does not mean that no use can be made of the inherent small parameter. The local nature of the non-uniformity itself gives often the possibility of another reduction. In such a case we call this a couple of limiting forms, “inner and outer problems”, and are evidence of the fact that we have apparently physically two connected but different problems as far as the dominating mechanism is concerned. Depending on the problem, we now have two simpler problems, serving as boundary conditions to each other via continuity or *matching* conditions.

6.1.2.1 Non-uniform asymptotic approximations

If a function of x and ε is "essentially" (we will see later what that means) dependent of a combination like x/ε (or anything equivalent, like $(x-x_0)/\varepsilon^2$), then there exists no regular (that means: uniform) asymptotic expansion for all $x = O(1)$ considered. A different expansion arises when $x = O(\varepsilon)$, in other words after scaling $t = x/\varepsilon$ where $t = O(1)$. If the limit exists, we may see something like

$$\Phi(x,\varepsilon) = \varphi\left(\frac{x}{\varepsilon}, x, \varepsilon\right) \simeq \varphi(\infty, x, 0) + \ldots, \quad \Phi(\varepsilon t, \varepsilon) = \varphi(t, \varepsilon t, \varepsilon) \simeq \varphi(t, 0, 0) + \ldots.$$

where x is assumed fixed and non-zero.

Practical examples are

$$\begin{aligned}
\mathrm{e}^{-x/\varepsilon} + \sin(x+\varepsilon) &= 0 + \sin x + \varepsilon \cos x + \ldots \quad \text{on } x \in (0,\infty) \\
\mathrm{e}^{-t} + \sin(\varepsilon t + \varepsilon) &= \mathrm{e}^{-t} + \varepsilon(t+1) + \ldots \quad \text{on } t \in [0, L] \\
\arctan\left(\frac{x}{\varepsilon}\right) + \tan(\varepsilon x) &= \frac{\pi}{2} + \varepsilon\left(x - \frac{1}{x}\right) + \ldots \quad \text{on } x \in (0,\infty) \\
\arctan(t) + \tan(\varepsilon^2 t) &= \arctan(t) + \varepsilon^2 t + \ldots \quad \text{on } t \in [0, L] \\
\frac{1}{x^2+\varepsilon^2} &= \frac{1}{x^2} - \frac{\varepsilon^2}{x^4} + \ldots \quad \text{on } x \in (0,\infty) \\
\frac{1}{\varepsilon^2 t^2 + \varepsilon^2} &= \varepsilon^{-2}\frac{1}{1+t^2} \quad \text{on } t \in [0, L]
\end{aligned}$$

where L is some constant. Of course, if x occurs only in a combination like x/ε, the asymptotic approximation becomes trivial after transformation, but that is only here for the example.

We call this expansion the *outer expansion*, principally valid in the "$x = O(1)$"-outer region. Now consider the *stretched coordinate*

$$t = \frac{x}{\varepsilon}.$$

If the transformed $\Psi(t,\varepsilon) = \Phi(x,\varepsilon)$ has a non-trivial regular asymptotic expansion, then we call this expansion the *inner expansion*, principally valid in the "$t = O(1)$"-inner region, or *boundary layer*. The adjective "non-trivial" is essential: the expansion must be *significant*, *i.e.* different from the outer-expansion rewritten in the inner variable t. This determines the choice (in the present examples) of the inner variable $t = x/\varepsilon$. The scaling $\delta(\varepsilon) = \varepsilon$ is the asymptotically *largest* gauge function with this property.

Note the following example, where we have *three* inherent length scales: $x = O(1)$, $x = O(\varepsilon)$, $x = O(\varepsilon^2)$ and therefore two (nested) boundary layers $x = \varepsilon t$ and $x = \varepsilon^2 \tau$,

$$\begin{aligned}
\log(x/\varepsilon + \varepsilon) &= -\log(\varepsilon) + \log(x) + \ldots\ldots \quad \text{on } x \in (0,\infty) \\
\log(t+\varepsilon) &= \log(t) + \frac{\varepsilon}{t} + \ldots \quad \text{on } t \in (0, L] \\
\log(\varepsilon\tau + \varepsilon) &= \log(\varepsilon) + \log(\tau + 1) \quad \text{on } \tau \in [0, L]
\end{aligned}$$

An important relation between an inner and an outer expansion is the hypothesis that they *match*: the respective regions of validity should, asymptotically, overlap (known as the *overlap hypothesis*). Algorithmically, one may express this as follows, known as Van Dyke's Rule. *The outer limit of the* *inner expansion should be equal to the inner limit of the outer expansion.* In other words, the outer-expansion, rewritten in the inner-variable, has a regular series expansion, which is *equal* to the regular asymptotic expansion of the inner-expansion, rewritten in the outer-variable.

Suppose that we have an outer expansion $\mu_0\phi_0 + \mu_1\phi_1 + \dots$ in outer variable x and a corresponding inner expansion $\lambda_0\psi_0 + \lambda_1\psi_1 + \dots$ in inner variable t, where $x = \delta t$. Suppose we can re-expand the outer expansion in the inner variable and the inner expansion in the outer variable such that

$$\begin{aligned} \mu_0(\varepsilon)\varphi_0(\delta t) \quad + \mu_1(\varepsilon)\varphi_1(\delta t) \quad + \dots &= \lambda_0(\varepsilon)\eta_0(t) \; + \lambda_1(\varepsilon)\eta_1(t) \; + \dots , \\ \lambda_0(\varepsilon)\psi_0(x/\delta) + \lambda_1(\varepsilon)\psi_1(x/\delta) + \dots &= \mu_0(\varepsilon)\theta_0(x) + \mu_1(\varepsilon)\theta_1(x) + \dots , \end{aligned}$$

Then for matching the results should be equivalent to the order considered. In particular the expansion of η_k, written back in x,

$$\lambda_0(\varepsilon)\eta_0(x/\delta) + \lambda_1(\varepsilon)\eta_1(x/\delta) + \dots = \mu_0(\varepsilon)\zeta_0(x) + \mu_1(\varepsilon)\zeta_1(x) + \dots,$$

should be such that $\zeta_k = \theta_k$ for $k = 0, 1, \cdots$.

A simple but typical example is the following function on $x \in [0, \infty)$

$$f(x, \varepsilon) = \sin(x + \varepsilon) + \mathrm{e}^{-x/\varepsilon} \cos x$$

with outer expansion with $x = O(1)$

$$F(x, \varepsilon) = \sin x + \varepsilon \cos x - \tfrac{1}{2}\varepsilon^2 \sin x - \tfrac{1}{6}\varepsilon^3 \cos x + O(\varepsilon^4)$$

and inner expansion with boundary layer (*i.e.* inner) variable $t = x/\varepsilon = O(1)$

$$G(t, \varepsilon) = \mathrm{e}^{-t} + \varepsilon(t + 1) - \tfrac{1}{2}\varepsilon^2 t^2\, \mathrm{e}^{-t} - \tfrac{1}{6}\varepsilon^3(t + 1)^3 + O(\varepsilon^4).$$

The outer expansion in the inner variable

$$F(\varepsilon t, \varepsilon) = \sin(\varepsilon t) + \varepsilon \cos(\varepsilon t) - \tfrac{1}{2}\varepsilon^2 \sin(\varepsilon t) - \tfrac{1}{6}\varepsilon^3 \cos(\varepsilon t) + O(\varepsilon^4)$$

becomes re-expanded

$$F_{\mathrm{in}}(t, \varepsilon) = \varepsilon(t + 1) - \tfrac{1}{6}\varepsilon^3(t + 1)^3 + O(\varepsilon^4)$$

which is, rewritten in x (and re-ordered in powers of ε), given by

$$F_{\mathrm{in}}(x/\varepsilon, \varepsilon) = x - \tfrac{1}{6}x^3 + \varepsilon(1 - \tfrac{1}{2}x^2) - \tfrac{1}{2}\varepsilon^2 x - \tfrac{1}{6}\varepsilon^3 + O(\varepsilon^4).$$

The inner expansion in the outer variable

$$G(x/\varepsilon, \varepsilon) = x + \varepsilon + (1 - \tfrac{1}{2}x^2)\, \mathrm{e}^{-x/\varepsilon} - \tfrac{1}{6}(x + \varepsilon)^3 + O(\varepsilon^4)$$

becomes re-expanded

$$G_{\mathrm{out}}(x, \varepsilon) = x - \tfrac{1}{6}x^3 + \varepsilon(1 - \tfrac{1}{2}x^2) - \tfrac{1}{2}\varepsilon^2 x - \tfrac{1}{6}\varepsilon^3 + O(\varepsilon^4).$$

Indeed is $G_{\mathrm{out}}(x, \varepsilon)$ functionally equal to $F_{\mathrm{in}}(x/\varepsilon, \varepsilon)$ to the order considered.

Another way to present matching is via an intermediate scaling. Conceptually, this remains closer to the idea of overlapping expansions than Van Dyke's matching rule, but in practice it is more laborious. Suppose we have an outer expansion $F(x, \varepsilon)$ in the outer variable x, and a corresponding inner expansion $G(t, \varepsilon)$ in the boundary layer variable t, where $x = \delta t$ and $\delta(\varepsilon) = o(1)$. Then for matching there should be an intermediate scaling $x = \eta\xi$, with[1] $\delta \ll \eta \ll 1$, such that under this scaling, the re-expanded outer expansion $[F(\eta\xi,\varepsilon)]_{\text{exp}}$ is equal (to the orders considered) to the re-expanded inner expansion $[G(\frac{\eta}{\delta}\xi,\varepsilon)]_{\text{exp}}$. Note that the result must not depend on the exact choice of η, and the expansions should be taken of high enough order.

With the above example we have with (for exampe) $\eta \sim \varepsilon^{\frac{1}{2}}$

$$\begin{aligned}F(\eta\xi,\varepsilon) &= \sin(\eta\xi) + \varepsilon\cos(\eta\xi) - \tfrac{1}{2}\varepsilon^2\sin(\eta\xi) - \tfrac{1}{6}\varepsilon^3\cos(\eta\xi) + O(\varepsilon^4)\\ &= \eta\xi + \varepsilon - \tfrac{1}{6}\eta^3\xi^3 - \tfrac{1}{2}\varepsilon\eta^2\xi^2 - \tfrac{1}{2}\varepsilon^2\eta\xi - \tfrac{1}{6}\varepsilon^3 + \tfrac{1}{120}\eta^5\xi^5 + \dots\end{aligned}$$

which is indeed to leading orders equal to

$$\begin{aligned}G(\tfrac{\eta}{\varepsilon}\xi,\varepsilon) &= \mathrm{e}^{-\eta\xi/\varepsilon} + \varepsilon(\tfrac{\eta}{\varepsilon}\xi + 1) - \tfrac{1}{2}\varepsilon^2(\tfrac{\eta}{\varepsilon}\xi)^2\,\mathrm{e}^{-\eta\xi/\varepsilon} - \tfrac{1}{6}\varepsilon^3(\tfrac{\eta}{\varepsilon}\xi + 1)^3 + O(\varepsilon^4).\\ &= \eta\xi + \varepsilon - \tfrac{1}{6}(\eta\xi + \varepsilon)^3 + \dots.\end{aligned}$$

The idea of matching is very important because it allows one to move smoothly from one regime into the other. The method of constructing local, but matching, expansions is therefore called "Matched Asymptotic Expansions" (MAE). An intermediate variable is typically used in evaluating integrals across a boundary layer (see below).

6.1.2.2 Constructing asymptotic solutions

The most important application of this concept of inner- and outer-expansions is that approximate solutions of certain differential equations can be constructed for which the limit under a small parameter is apparently non-uniform.

The main lines of argument for constructing a MAE solution to a differential equation satisfying some boundary conditions are as follows. Suppose we have the following (example) problem.

$$\varepsilon\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{\mathrm{d}\varphi}{\mathrm{d}x} - 2x = 0, \qquad \varphi(0) = \varphi(1) = 2. \tag{6.1}$$

Assuming that the outer solution is $O(1)$ because of the boundary conditions, we have for the equation to leading order

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}x} - 2x = 0,$$

with solution

$$\varphi_0 = x^2 + A.$$

The integration constant A can be determined by the boundary condition $\varphi_0(0) = 2$ at $x = 0$ or $\varphi_0(1) = 2$ at $x = 1$, but not both, so we expect a boundary layer at either end. By trial and error we find that no solution can be constructed if we assume a boundary layer at $x = 1$, so, inferring a boundary layer at $x = 0$, we have to use the boundary condition at $x = 1$ and find

$$\varphi_0 = x^2 + 1.$$

[1] In other words, $\delta = o(\eta)$ and $\eta = o(1)$.

The structure of the equation indeed suggests a correction of $O(\varepsilon)$, so we try the expansion

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots .$$

For φ_1 this results into the equation

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}x} + \frac{\mathrm{d}^2\varphi_0}{\mathrm{d}x^2} = 0,$$

with $\varphi_1(1) = 0$ (the $O(\varepsilon)$-term of the boundary condition), which has the solution

$$\varphi_1 = 2 - 2x.$$

Higher orders are straightforward:

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}x} = 0, \qquad \text{with } \varphi_n(1) = 0,$$

leading to solutions $\varphi_n \equiv 0$. We find for the outer expansion

$$\varphi = x^2 + 1 + 2\varepsilon(1-x) + O(\varepsilon^N). \tag{6.2}$$

We continue with the inner expansion, and find near $x = 0$, an order of magnitude of the solution givne by $\varphi = \lambda\psi$, and a boundary layer thickness given by $x = \delta t$ (both λ and δ are to be determined)

$$\frac{\varepsilon\lambda}{\delta^2}\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + \frac{\lambda}{\delta}\frac{\mathrm{d}\psi}{\mathrm{d}t} - 2\delta t = 0.$$

Both from the matching ($\varphi_{\text{outer}} \to 1$ for $x \downarrow 0$) and from the boundary condition ($\varphi(0) = 2$) we have to conclude that $\varphi_{\text{inner}} = O(1)$ and so $\lambda = 1$. Furthermore, the boundary layer has only a reason for existence if it comprises new effects, not described by the outer solution. From the heuristic *correspondence principle* we expect that (*i*) a meaningful rescaling corresponds with a *distinguished limit* or *significant degeneration*, while (*ii*) new effects are only included if we have a new equation; in this case if $(\mathrm{d}^2\psi/\mathrm{d}t^2)$ is included. So $\varepsilon\delta^{-2}$ must be at least as large as δ^{-1}, the largest of δ^{-1} and δ. From the principle that we look for the equation with the richest structure, it must be exactly as large, implying a boundary layer thickness $\delta = \varepsilon$. Thus we have the inner equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi}{\mathrm{d}t} - 2\varepsilon^2 t = 0.$$

From this equation it would *seem* that we have a series expansion without the $O(\varepsilon)$-term, since the equation for this order would be the same as for the leading order. However, from matching with the outer solution:

$$\varphi_{\text{outer}} \to 1 + 2\varepsilon + \varepsilon^2(t^2 - 2t) + \cdots \qquad (x = \varepsilon t,\ t = O(1)),$$

we see that an additional $O(\varepsilon)$-term is to be included. So we substitute the series expansion:

$$\varphi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots \tag{6.3}$$

It is a simple matter to find

$$\begin{aligned}
\frac{\mathrm{d}^2\psi_0}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi_0}{\mathrm{d}t} &= 0, \quad \psi_0(0) = 2 \ \longrightarrow\ \psi_0 = 2 + A_0(\mathrm{e}^{-t} - 1),\\
\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi_1}{\mathrm{d}t} &= 0, \quad \psi_1(0) = 0 \ \longrightarrow\ \psi_1 = A_1(\mathrm{e}^{-t} - 1),\\
\frac{\mathrm{d}^2\psi_2}{\mathrm{d}t^2} + \frac{\mathrm{d}\psi_2}{\mathrm{d}t} &= 2t, \quad \psi_2(0) = 0 \ \longrightarrow\ \psi_2 = t^2 - 2t + A_2(\mathrm{e}^{-t} - 1),
\end{aligned}$$

where the constants $A_0, A_1, A_2, \cdots$ are to be determined from the matching condition that inner and outer solution should be asymptotically equivalent in the region of overlap. We follow Van Dyke's matching rule, and rewrite outer expansion (6.2) in inner variable t, inner expansion (6.3) in outer variable x, re-expand and rewrite the result in x. This results into

$$x^2 + 1 + 2\varepsilon(1 - x) + O(\varepsilon^3) \simeq 1 + 2\varepsilon + x^2 - 2\varepsilon x + O(\varepsilon^3) \tag{6.4a}$$

$$\begin{aligned} &2 + A_0(\mathrm{e}^{-t} - 1) + \varepsilon A_1(\mathrm{e}^{-t} - 1) + \varepsilon^2\big(t^2 - 2t + A_2(\mathrm{e}^{-t} - 1)\big) + O(\varepsilon^3) \\ &\qquad \simeq 2 - A_0 - \varepsilon A_1 + x^2 - 2\varepsilon x - \varepsilon^2 A_2 + O(\varepsilon^3) \end{aligned} \tag{6.4b}$$

The resulting reduced expressions (6.4a) and (6.4b) must be functionally equivalent. A full matching is thus obtained if we choose $A_0 = 1$, $A_1 = -2$, $A_2 = 0$.

6.1.2.3 Composite expansion

If the boundary layer structure is simple enough, in particular if we have just a simple boundary layer with matching inner and outer expansions, it is possible to combine the separate expansions into a single uniform expansion, called a composite expansion.

Suppose we have an outer expansion $\phi = \mu_0\phi_0 + \mu_1\phi_1 + \ldots$ in outer variable $x \in (0, 1)$ and a corresponding inner expansion $\psi = \lambda_0\psi_0 + \lambda_1\psi_1 + \ldots$ in inner variable $t \in [0, \infty)$, where $x = \delta t$ and $\delta(\varepsilon) = o(1)$. In view of matching, the overlapping parts

$$\begin{aligned} \hat{\phi}(x) &= \big[\phi(\delta t)\big]_{t=x/\delta} = \big[\mu_0(\varepsilon)\varphi_0(\delta t) + \mu_1(\varepsilon)\varphi_1(\delta t) + \ldots\big]_{t=x/\delta} = \big[\lambda_0(\varepsilon)\eta_0(t) + \lambda_1(\varepsilon)\eta_1(t) + \ldots\big]_{t=x/\delta} \\ \hat{\psi}(x) &= \psi(x/\delta) = \lambda_0(\varepsilon)\psi_0(x/\delta) + \lambda_1(\varepsilon)\psi_1(x/\delta) + \ldots = \mu_0(\varepsilon)\theta_0(x) + \mu_1(\varepsilon)\theta_1(x) + \ldots \end{aligned}$$

are functionally equivalent to the order considered, *i.e.* $\hat{\phi} \simeq \hat{\psi}(x)$. This means that the combined expression

$$\phi(x) + \psi(x/\delta)$$

is for $x = O(1)$ asymptotically equal to $\phi(x) + \hat{\psi}(x)$, and for $x = O(\delta)$ asymptotically equal to $\psi(x) + \hat{\phi}(x)$. In both cases it is the overlapping part $\hat{\phi}(x)$ (or equivalently $\hat{\psi}(x)$) which is too much. The combined expansion

$$\Phi(x) = \phi(x) + \psi(x/\delta) - \hat{\phi}(x)$$

is thus valid both in the boundary layer and in the outer region.

As an example we may consider the previous problem (6.1), with solution (reformulated)

$$\begin{aligned} \phi(x) &= x^2 + 1 + 2\varepsilon(1 - x) + O(\varepsilon^3) \\ \psi(t) &= 1 + \mathrm{e}^{-t} - 2\varepsilon(\mathrm{e}^{-t} - 1) + \varepsilon^2(t^2 - 2t) + O(\varepsilon^3) \\ \hat{\phi}(x) &= 1 + 2\varepsilon + x^2 - 2\varepsilon x + O(\varepsilon^3) \\ \Phi(x) &= x^2 + 1 + \mathrm{e}^{-x/\varepsilon} + 2\varepsilon - 2\varepsilon x - 2\varepsilon\,\mathrm{e}^{-x/\varepsilon} + O(\varepsilon^3) \end{aligned}$$

6.1.2.4 Approximate evaluation of integrals

Another application of MAE is integration. We split the integral halfway the region of overlap, and approximate the integrand by its inner and outer approximation. Take for example

$$f(x,\varepsilon)=\frac{\log(1+x)}{x^2+\varepsilon^2},\quad 0\leqslant x<\infty,\quad 0<\varepsilon\ll 1,$$

with outer expansion

$$f(x,\varepsilon)=\frac{\log(1+x)}{x^2+\varepsilon^2}=\frac{\log(1+x)}{x^2}-\varepsilon^2\frac{\log(1+x)}{x^4}+O(\varepsilon^4)$$

and inner expansion in boundary layer $x=\varepsilon t$

$$f(\varepsilon t,\varepsilon)=\frac{\log(1+\varepsilon t)}{\varepsilon^2(t^2+1)}=\frac{1}{\varepsilon^2}\Big(\frac{\varepsilon t-\frac12\varepsilon^2t^2+O(\varepsilon^3)}{t^2+1}\Big)=\frac{1}{\varepsilon}\frac{t}{t^2+1}-\frac{\frac12 t^2}{t^2+1}+O(\varepsilon).$$

If we introduce a function $\eta=\eta(\varepsilon)$ with $\varepsilon\ll\eta\ll 1$ (note that eventually the detailed choice of η is and should be immaterial), and split up the integration interval $[0,\infty)=[0,\eta]\cup[\eta,\infty)$, we find

$$\begin{aligned}\int_0^\infty \frac{\log(1+x)}{x^2+\varepsilon^2}\,\mathrm{d}x &\simeq \int_0^{\eta/\varepsilon}\frac{t}{t^2+1}+O(\varepsilon)\,\mathrm{d}t+\int_\eta^\infty\frac{\log(1+x)}{x^2}+O(\varepsilon^2)\,\mathrm{d}x\\ &=\Big[\tfrac12\log(1+t^2)+O(\varepsilon)\Big]_0^{\eta/\varepsilon}+\Big[\log x-\frac{1+x}{x}\log(1+x)+O(\varepsilon^2/x^2)\Big]_\eta^\infty\\ &=\Big(\log\eta+\tfrac12\log(1+\varepsilon^2/\eta^2)-\log\varepsilon+O(\eta)\Big)+\Big(\log(1+\eta)-\log\eta+\frac{\log(1+\eta)}{\eta}+O(\varepsilon^2/\eta^2)\Big)\\ &\simeq-\log\varepsilon+1.\end{aligned}$$

6.1.2.5 Implicit matching subtleties

An interesting detail in the matching process of boundary layer problems where the inner equation is a form of Newton's equation (for example exercises 6.2.11, 6.2.12, 6.2.23, and others) is the following. Consider a boundary layer equation in $Y(t)=Y_0(t)+\ldots$, $0\leqslant t<\infty$, of the form

$$\tfrac{\partial^2}{\partial t^2}Y_0+F'(Y_0)=0,$$

which may be integrated to

$$\tfrac12(\tfrac{\partial}{\partial t}Y_0)^2+F(Y_0)=E.$$

If Y_0 should be matched for $t\to\infty$ to an outer solution $y(x)$ of $O(1)$ with $x=\varepsilon t$, then the integration constant E may be found by observing that $y_x\sim\varepsilon^{-1}Y_t=O(1)$, so the leading order Y_{0t} should vanish for large t. Hence $E=F(y(0))$. An important condition for consistency is that the final integral

$$\int_{Y_0(0)}^{Y_0}\frac{1}{\sqrt{E-F(\eta)}}\,\mathrm{d}\eta=\pm\sqrt{2}t$$

diverges (no square root singularity but at least a simple pole) at $\eta=y(0)$, in order to have $t\to\infty$.

We illustrate this by the following example. The singular boundary value problem

$$\varepsilon^2 y'' + y^2 = K(x), \quad y(0) = 0, \quad y(1) = 0$$

where $K(x) > c > 0$ is $O(1)$ and sufficiently smooth, has boundary layers of $O(\varepsilon)$ near $x = 0$ and $x = 1$. We consider $x = 0$. (The other is analogous.)

An outer approximation $y = y_0 + \dots$ is readily found to be

$$y_0(x) = \pm\sqrt{K(x)},$$

with sign to be decided. Write for notational convenience $K(0) = k^2$. The leading order inner equation for $y(x) = Y(t) = Y_0 + \dots$, where $x = \varepsilon t$, is

$$Y_0'' + Y_0^2 = k^2, \quad Y_0(0) = 0.$$

As argued above, for matching it is required that $Y_0(t) \to \pm k$ and $Y_0' \to 0$. We integrate

$$\tfrac{1}{2}(Y_0')^2 + \tfrac{1}{3}Y_0^3 - k^2 Y_0 = E = \pm(\tfrac{1}{3}k^3 - k^3) = \tfrac{2}{3}k^3.$$

Since Y_0 is small for $t \to 0$ and $(Y_0')^2 > 0$, the sign of E can only be positive, and *thus* outer solution $y_0(x)$ must be negative. Furthermore

$$\tfrac{2}{3}k^3 - \tfrac{1}{3}Y_0^3 + k^2 Y_0 = \tfrac{1}{3}(Y_0 + k)^2(2k - Y_0).$$

Noting that Y_0' has to be negative, we can finish as usual to find explicitly

$$\int_{Y_0}^{0} \frac{1}{(\eta + k)\sqrt{2k - \eta}}\,\mathrm{d}\eta = \tfrac{2}{3}\sqrt{3}\left[\operatorname{artanh}\left(\sqrt{\frac{2k - Y_0}{3k}}\right) - \operatorname{artanh}\left(\sqrt{\tfrac{2}{3}}\right)\right] = \sqrt{\tfrac{2}{3}k}\,t$$

such that

$$Y_0(t) = 2k - 3k\tanh^2\left(\sqrt{\tfrac{1}{2}k}\,t + \operatorname{artanh}\left(\sqrt{\tfrac{2}{3}}\right)\right)$$

6.1.2.6 Logarithmic switchback

It is not always evident from just the structure of the equation what the necessary expansion will look like. Sometimes it is well concealed and we are only made aware of an invalid initial choice by a matching failure. In fact, it is also the matching process itself that reveals us the required sequence of scaling functions. An example of such a back reaction is known as *logarithmic switchback*.

Consider the following problem for $y = y(x, \varepsilon)$ on the unit interval.

$$\varepsilon y'' + x(y' - y) = 0, \ 0 < x < 1, \qquad y(0, \varepsilon) = 0, \quad y(1, \varepsilon) = \mathrm{e}.$$

The outer solution appears to have the expansion

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + O(\varepsilon^3).$$

By trial and error, the boundary layer appears to be located near $x = 0$, so the governing equations and boundary conditions are then

$$\begin{aligned} &y_0' - y_0 = 0, && y_0(1) = \mathrm{e}, \\ &y_n' - y_n = -x^{-1} y_{n-1}'', && y_n(1) = 0, \end{aligned}$$

with general solution

$$y_n(x) = A_n\,\mathrm{e}^x + \int_x^1 z^{-1}\,\mathrm{e}^{x-z}\,y_{n-1}''(z)\,\mathrm{d}z,$$

such that

$$\begin{aligned} y_0(x) &= \mathrm{e}^x, \\ y_1(x) &= -\,\mathrm{e}^x \ln(x), \\ y_2(x) &= \mathrm{e}^x \big(\tfrac{1}{2}\ln(x)^2 + \tfrac{3}{2} - 2x^{-1} + \tfrac{1}{2}x^{-2}\big), \end{aligned}$$

etc. The boundary layer thickness is found from the assumed scaling $x = \varepsilon^m t$ and noting that $y = O(1)$ because of the matching with the outer solution. This leads to the significant degeneration of $m = \frac{1}{2}$, or $x = \varepsilon^{\frac{1}{2}} t$. The boundary layer equation for $y(x, \varepsilon) = Y(t, \varepsilon)$ is thus

$$Y'' + tY' - \varepsilon^{\frac{1}{2}} tY = 0, \qquad Y(0, \varepsilon) = 0.$$

The obvious choice of expansion of Y in powers of $\varepsilon^{\frac{1}{2}}$ is not correct, as the found solution does not match with the outer solution. Therefore, we consider the outer solution in more detail for small x. When $x = \varepsilon^{\frac{1}{2}} t$, we have for the outer solution

$$y(\varepsilon^{\frac{1}{2}} t, \varepsilon) = 1 + \varepsilon^{\frac{1}{2}} t + \varepsilon\big(-\tfrac{1}{2}\ln\varepsilon + \tfrac{1}{2}t^2 - \ln t + \tfrac{1}{2}t^{-2} + \dots\big) + O(\varepsilon^{\frac{3}{2}}\ln\varepsilon) \tag{6.5}$$

(The dots indicate powers of t^{-2} that appear with higher order y_n.) So we apparently need at least

$$Y(t, \varepsilon) = Y_0(t) + \varepsilon^{\frac{1}{2}} Y_1(t) + \varepsilon\ln(\varepsilon) Y_2(t) + \varepsilon Y_3(t) + o(\varepsilon),$$

with equations and boundary conditions

$$\begin{aligned} &Y_0'' + tY_0' = 0, && Y_0(0) = 0, \\ &Y_1'' + tY_1' = tY_0, && Y_1(0) = 0, \\ &Y_2'' + tY_2' = 0, && Y_2(0) = 0, \\ &Y_3'' + tY_3' = tY_1, && Y_3(0) = 0, \end{aligned}$$

etc. Hence, the inner expansion is given by

$$\begin{aligned} Y_0(t) &= A_0\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big), \\ Y_1(t) &= A_1\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big) + A_0\big[t\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big) + 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}}(\mathrm{e}^{-\frac{1}{2}t^2} - 1)\big], \\ Y_2(t) &= A_2\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big), \\ Y_3(t) &= A_3\,\mathrm{erf}\big(\tfrac{t}{\sqrt{2}}\big) + \int_0^t \mathrm{e}^{-\frac{1}{2}z^2} \int_0^z \mathrm{e}^{\frac{1}{2}\xi^2}\,\xi Y_1(\xi)\,\mathrm{d}\xi\,\mathrm{d}z. \end{aligned}$$

Unfortunately, Y_3 cannot be expressed in closed form. However, for demonstration it is sufficient to derive the behaviour of Y_3 for large t. As $\operatorname{erf}(z) \to 1$ exponentially fast for $z \to \infty$, we obtain

$$Y_1(t) = A_0 t + A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} A_0 + \text{ exponentially small terms.}$$

If Y_3 behaves for large t algebraically, then $tY_3' \gg Y_3''$, so $Y_3' = Y_1 - t^{-1}Y_3'' \simeq A_0 t$. By successive substitution it follows that

$$Y_3(t) = \tfrac{1}{2}A_0 t^2 + (A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} A_0)t - A_0 \ln(t) + \dots$$

For matching of the inner solution, we introduce the intermediate variable $\eta = \varepsilon^{-\alpha}x = \varepsilon^{\frac{1}{2}-\alpha}t$ where $0 < \alpha < \frac{1}{2}$, and compare with expression (6.5). We have

$$\begin{aligned}
A_0 + \varepsilon^{\frac{1}{2}}\big(A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} A_0\big) + \varepsilon^{\alpha} A_0 \eta + \varepsilon \ln(\varepsilon) A_2 + \tfrac{1}{2}\varepsilon^{2\alpha} A_0 \eta^2 \\
+ \varepsilon^{\frac{1}{2}+\alpha}\big(A_1 - 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}} A_0)\eta - \varepsilon A_0 \ln \eta + \varepsilon(\tfrac{1}{2} - \alpha) A_0 \ln \varepsilon \\
\equiv 1 + \varepsilon^{\alpha}\eta + \tfrac{1}{2}\varepsilon^{2\alpha}\eta^2 - \varepsilon \ln \eta - \alpha\varepsilon \ln(\varepsilon) + \tfrac{1}{2}\varepsilon^{2-2\alpha}\eta^{-2}.
\end{aligned}$$

Noting that $2 - 2\alpha > 1$, we find a full matching with

$$A_0 = 1, \quad A_1 = 2\big(\tfrac{2}{\pi}\big)^{\frac{1}{2}}, \quad A_2 = -\tfrac{1}{2}.$$

This problem is an example where intermediate matching is preferable.

6.1.2.7 Prandtl's boundary layer analysis.

The start of modern boundary layer theory is Prandtl's analysis in 1904 of the canonical problem of uniform incompressible low-viscous flow of main flow speed U_∞, viscosity μ and density ρ_0, along a flat plate of length L. Consider the stationary 2D Navier-Stokes equations for incompressible flow for velocity (u, v) and pressure p

$$\begin{aligned}
u_x + v_y &= 0, \\
\rho_0(uu_x + vu_y) &= -p_x + \mu(u_{xx} + u_{yy}), \\
\rho_0(uv_x + vv_y) &= -p_y + \mu(v_{xx} + v_{yy}),
\end{aligned}$$

subject to boundary conditions $u = v = 0$ at $y = 0$, $0 < x < L$.

Make dimensionless $u := U_\infty u$, $v := U_\infty v$, $p := \rho_0 U_\infty^2 p$, $x := Lx$, $y := Ly$. (The scaling of the pressure may not be evident, but is due to the fact that the low-viscous problem is inertia dominated, so the pressure gradient, which is really a reaction force, should balance the inertia terms.) We are left with the dimensionless *Reynolds number* $Re = \rho_0 U_\infty L/\mu$. Since Re is supposed to be large, we write $\varepsilon = Re^{-1}$ small. We obtain

$$\begin{aligned}
u_x + v_y &= 0, \\
uu_x + vu_y &= -p_x + \varepsilon(u_{xx} + u_{yy}), \\
uv_x + vv_y &= -p_y + \varepsilon(v_{xx} + v_{yy}),
\end{aligned}$$

subject to boundary conditions $u = v = 0$ at $y = 0$, $0 < x < 1$.

The leading order outer solution for $y = O(1)$ is given by

$$(u, v, p) = (1, 0, 0)$$

but this solution does not satisfy the boundary condition $u = 0$ at $y = 0$ along the plate. So we anticipate a boundary layer in y, such that the viscous friction εu_{yy} contributes. When we scale $x = X$, $y = \varepsilon^n Y$, $u = U$, $v = \varepsilon^m V$, and $p = P$, we find

$$\begin{aligned} U_X + \varepsilon^{m-n} V_Y &= 0, \\ UU_X + \varepsilon^{m-n} VU_Y &= -P_X + \varepsilon U_{XX} + \varepsilon^{1-2n} U_{YY}, \\ \varepsilon^m UV_X + \varepsilon^{2m-n} VV_Y &= -\varepsilon^{-n} P_Y + \varepsilon^{1+m} V_{XX} + \varepsilon^{1+m-2n} V_{YY}. \end{aligned}$$

This yields the distinguished limit $m = n = \frac{1}{2}$, with the significant degeneration

$$\begin{aligned} U_X + V_Y &= 0, \\ UU_X + VU_Y &= U_{YY}, \\ P_Y &= 0, \end{aligned}$$

known as *Prandtl's Boundary Layer Equations*. Since $P = P(X)$ has to match to the outer solution $p =$ constant (for this particular flat plate problem), pressure gradient $P_X = 0$ and disappears to leading order.

Very quickly after Prandtl's introduction of his boundary layer equations, Blasius (1906) was able to reduce the equation to an ordinary differential equation by means of a similarity solution for the stream function ψ, with $U = \psi_Y$ and $V = -\psi_X$, of the form

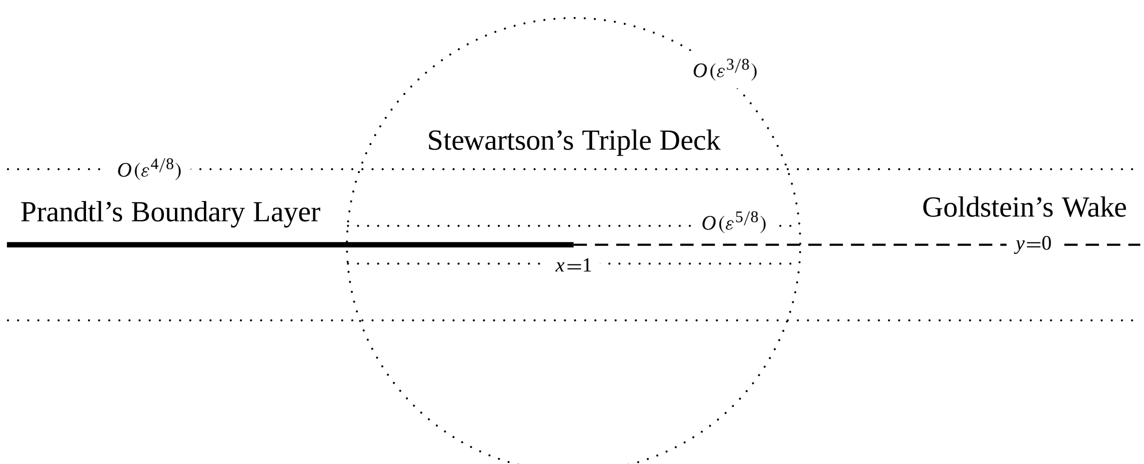
$$\psi(X, Y) = \sqrt{2X}\, f(\eta), \qquad \eta = \frac{Y}{\sqrt{2X}},$$

leading to Blasius' equation

$$f''' + ff'' = 0.$$

Prandtl's boundary layer equations, but with other boundary conditions, are also valid in the viscous wake behind the plate $x > 1$, $y = O(\varepsilon^{1/2})$ (Goldstein, 1930).

The trailing edge region around $x = 1$, $y = 0$, however, is far more complicated (Stewartson, 1969). Here the boundary layer structure consists of three layers: $y = O(\varepsilon^{5/8})$, $O(\varepsilon^{4/8})$, $O(\varepsilon^{3/8})$ within $x - 1 = O(\varepsilon^{3/8})$. This is known as Stewartson's Triple Deck.



6.1.2.8 The rôle of matching

It is important to note that a matching is possible at all! Only a part of the terms can be matched by selection of the undetermined constants. Other terms are already equal, without free constants, and there is no way to repair a possibly incomplete matching here. This is an important consistency check on the found solution, at least as long as no real proof is available. If no matching appears to be possible, almost certainly one of the assumptions made with the construction of the solution has to be reconsidered. Particularly notorious are logarithmic singularities of the outer solution, as we saw above. See for other examples [13].

Summarizing, matching of inner- and outer expansion plays an important rôle in the following ways:

i) it provides information about the sequence of order (gauge) functions $\{\mu_k\}$ and $\{\lambda_k\}$ of the expansions;

ii) it allows us to determine unknown constants of integration;

iii) it provides a check on the consistency of the solution, giving us confidence in the correctness.

6.2 Matched Asymptotic Expansions: Assignments

6.2.1 Non-uniform approximations and boundary layers

Determine on $[0, \infty)$ the outer approximation and all inner approximations (the boundary layers, corresponding scaling and asymptotic expansions) of

$$\frac{1}{x + \varepsilon^2 + \varepsilon^3}$$

6.2.2 Boundary layers and integration

Consider the function

$$f(x, \varepsilon) = \mathrm{e}^{-x/\varepsilon}(1 + x) + \pi \cos(\pi x + \varepsilon) \quad \text{for} \quad 0 \leqslant x \leqslant 1.$$

a) Construct an outer and inner expansion of f with error $O(\varepsilon^3)$.

b) Integrate f from $x = 0$ to 1 exactly and expand the result up to $O(\varepsilon^3)$.

c) Compare this with the integral that is obtained by integration of the inner and outer expansions following the method described in section 6.1.2 (or Example 15.30 of SIAM book).

6.2.3 Friedrichs' model problem

Friedrichs' (1942) model problem for a boundary layer in a viscous fluid is

$$\varepsilon y'' = a - y' \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$, and a is a given positive constant of order 1 and independent of ε. Find a two-term inner and outer expansion of the solution of this problem.

6.2.4 Singularly perturbed ordinary differential equations

Determine the asymptotic approximation of solution $y(x, \varepsilon)$ (1st or 1st+2nd leading order terms for positive small parameter $\varepsilon \to 0$) of the following singularly perturbed problems.

α and β are non-zero constants, independent of ε.

Provide arguments for the determined boundary layer thickness and location, and show how free constants are determined by the matching procedure.

a)
$$\varepsilon y'' - y' = 2x, \qquad y(0, \varepsilon) = \alpha, \quad y(1, \varepsilon) = \beta.$$

b)
$$\varepsilon y' + y^2 = \cos(x), \qquad y(0, \varepsilon) = 0, \quad 0 \leqslant x \leqslant 1.$$

c)
$$\varepsilon y'' + (2x + 1)y' + y^2 = 0, \qquad y(0, \varepsilon) = \alpha, \quad y(1, \varepsilon) = \beta.$$

6.2.5 A hidden boundary layer structure

The problem for $\phi = \phi(x, \varepsilon)$ and $\varepsilon > 0$, $x \geqslant 0$

$$\varepsilon^2 \phi'' - F(x)\phi = 0, \qquad \phi(0) = a, \quad \phi'(0) = b$$

is difficult to analyse asymptotically for small ε (why?). We therefore transform the problem.

a) Rewrite the problem for $y(x)$ given by

$$\phi(x) = a \exp\left(\frac{1}{\varepsilon} \int_0^x y(z)\,\mathrm{d}z\right)$$

What is the initial condition?

b) Assume that F is sufficiently smooth (analytic), and $F(x) \geqslant c > 0$ along the interval of interest. Formulate a formal asymptotic solution of $y = y(x, \varepsilon)$ for small ε up to and including $O(\varepsilon)$.

c) Apply this to the asymptotic solution for $F(x) = \mathrm{e}^x$.

d) What changes when we apply the transformation

$$\phi(x) = a \exp\left(-\frac{1}{\varepsilon} \int_0^x y(z)\,\mathrm{d}z\right).$$

Explain why we obtain, in the end, the same result.

6.2.6 A singularly perturbed nonlinear problem

Find a composite expansion along $0 \leqslant x \leqslant 1$ of the solution of the following boundary value problem

$$\varepsilon y'' + 2y' + y^3 = 0, \qquad y(0) = 0, \quad y(1) = \tfrac{1}{2}.$$

6.2.7 A singularly perturbed linear problem

Find a composite expansion of the solution of the following boundary value problem along $0 < x < 1$

$$\varepsilon y'' = f(x) - y', \quad \text{where } y(0) = 0 \text{ and } y(1) = 1.$$

The function f is continuous, independent of ε and of order 1.

6.2.8 A boundary layer problem

(a) Show that the problem

$$\varepsilon x^m y' + y^2 = \cos x \quad \text{along} \quad x \in [0, 1], \quad y(0) = 0$$

with $0 < m < 1$, has a boundary layer near $x = 0$. Give the corresponding scaling of x.

(b) The same question if the right-hand side of the equation is $\sin x$.

6.2.9 Sign and scaling problems

A small parameter multiplying the highest derivative does not guarantee that boundary or interior layers are present. After solving the following problems directly, explain why the method of matched asymptotic expansions cannot be used (in a straightforward manner) to find an asymptotic approximation to the solution.

(a) $\varepsilon^2 y'' + \omega^2 y = 0$ along $0 < x < 1$ and $\omega \neq 0$.

(b) $\varepsilon y'' = y'$ along $0 < x < 1$, while $y'(0) = -1$ and $y(1) = 0$.

6.2.10 The Michaelis-Menten model

A classic enzyme-reaction model, for the first time proposed by Michaelis en Menten (1913), considers a substrate (concentration S) reacting with an enzyme (concentration E) to an enzyme-substrate complex (concentration C), that on its turn dissociates into the final product (concentration P) and the enzyme. The reaction of the substrate to the complex is described in time t by the system

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= -k_1 ES + k_{-1}C + k_2 C,\\
\frac{\mathrm{d}S}{\mathrm{d}t} &= -k_1 ES + k_{-1}C,\\
\frac{\mathrm{d}C}{\mathrm{d}t} &= \quad k_1 ES - k_{-1}C - k_2 C,\\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \quad k_2 C,
\end{aligned}$$

with initial values $S(0) = S_0$, $C(0) = 0$, $E(0) = E_0$ and $P(0) = 0$. The parameters k_1, k_{-1} and k_2 are reaction rates: k_1 of the forward reaction, k_{-1} of the backward reaction, and k_2 of the dissociation.

a. If $[S] = [C] = [E] = [P] = \mathrm{mol/m^3}$, and $[T] = \mathrm{s}$, what are the dimensional units of k_1, k_{-1} and k_2?

b. Expressed in de problem variables S, C, E, P en t, and the problem parameters E_0, S_0, k_1, k_{-1} and k_2, how many dimensionless quantities has this problem?
Note: “mol” is already dimensionless and does not count as separate unit.

c. Show that $E = E_0 - C$. Ignore the equation for P. Make S, C and t dimensionless such that we obtain a system of the form

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\tau} &= -s + sc + \lambda c,\\
\varepsilon\frac{\mathrm{d}c}{\mathrm{d}\tau} &= \quad s - sc - \mu c,
\end{aligned}$$

with $s(0) = 1$, $c(0) = 0$.

d. Consider the resulting problem asymptotically for $\varepsilon \to 0$. We see that there are two time scales (which?). The short one corresponds with the transient switch-on effects, which behave mathematically like a boundary layer in time. Solve the problem asymptotically to leading and first order in ε. *Hint:* it may be convenient to introduce the parameter $\nu = \mu - \lambda$.

6.2.11 Groundwater flow

Through a long strip of ground of width L between two canals (water level h_0 and h_1) the ground water seeps slowly from one side to the other.

Select a coordinate system such that the Z-axis is parallel to the long axis of the strip and the canals, the Y-axis is vertical, and the X-axis perpendicular to both. $X = 0$ corresponds with canal 0, and $X = L$ with canal 1. Assume that the groundwater level is constant in Z-direction.

Assume that the layer of ground lies on top of a semipermeable layer at level$Y = 0$, while the ground water level is given by $Y = h(X)$.

The water leaks through the semi-permeable layer at a rate proportional to the local hydrostatic pressure. As this pressure is on its turn proportional to water level h, this yields a flux density αh, where α is a constant.

Water comes in by precipitation (rain). Fluctuations in precipitation are assumed to be averaged away by the slow groundwater flow, such that the flux density N from this precipitation is constant in time. Assume that variations in overgrowth and buildings may result into a position dependent $N = N(X)$.

Between two neighbouring positions X and $X + \mathrm{d}X$ there exists a small difference in height and therefore in pressure. According to Darcy's law this creates a flow with a velocity proportional to the pressure difference, and dependent of the porosity of the ground. As the pressure difference is the same along the full height, the flow velocity is uniform, and we have

$$p(X) - p(X + \mathrm{d}X) \ \sim\ h(X) - h(X + \mathrm{d}X) \ \sim\ v(X)\,\mathrm{d}X,$$

and the horizontal flux density is proportional to

$$v = -D\frac{\mathrm{d}h}{\mathrm{d}X}$$

where D is in general a function of position.

The flux balance along a slice $\mathrm{d}X$ is then given by $\left[Dh\frac{\mathrm{d}h}{\mathrm{d}X}\right]_X^{X+\mathrm{d}X} = (\alpha h - N)\,\mathrm{d}X$, or

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(Dh\frac{\mathrm{d}h}{\mathrm{d}X}\right) = \alpha h - N$$

a. We consider the situation with $h_0 = 0$, and$D$ is constant. Make dimensionless with L, h_1 and α: $X = Lx$, $h(X) = h_1\phi(x)$, $N(X) = \alpha h_1 K(x)$, and introduce the positive dimensionless parameter
$$\varepsilon = \frac{Dh_1}{2\alpha L^2}.$$

b. Assume heavy rain, such that $K(x) = O(1)$. solve the resulting problem asymptotically for $\varepsilon \rightarrow 0$.

c. Assume little rain, such that $K(x) = \varepsilon\kappa(x)$, with $\kappa = O(1)$. Solve the resulting problem asymptotically for $\varepsilon \rightarrow 0$. Take good care at $x = 1$. The boundary layer is rather complicated with a layered structure.

d. What changes when we take the slightly more general case of $D = D(X)$?

6.2.12 Stirring a cup of tea

When we stir a cup of tea, the surface of the fluid deforms until equilibrium is attained between gravity, centrifugal force and surface tension. This last force is only important near the wall.

Consider for this problem the following model problem.

A cylinder (radius a, axis vertically) with fluid (density ρ, surface tension σ) rotates around its axis $\vec{e}_z$ (angular velocity Ω) in a gravity field $-g\vec{e}_z$. By the gravity and the centrifugal force the surface deforms to something that looks like a paraboloid. Within a small neighbourhood of the cylinder wall the contact angle α is felt by means of the surface tension.

Because of symmetry we can describe the surface by a radial tangent angle ψ with the horizon, parametrized by arc length s, such that $s = 0$ corresponds wit the axis, and $s = L$ with the wall of the cylinder. L is unknown.

Select the origin on the axis at the surface, such that he vertical and radial coordinate are given by

$$Z(s) = \int_0^s \sin\psi(s')\,\mathrm{d}s'$$
$$R(s) = \int_0^s \cos\psi(s')\,\mathrm{d}s'.$$

The necessary balance between hydrostatic pressure and surface tension yields the equation

$$p_0 - \rho g Z + \tfrac{1}{2}\rho\Omega^2 R^2 = -\sigma\left(\frac{\mathrm{d}\psi}{\mathrm{d}s} + \frac{\sin\psi}{R}\right)$$

with unknown p_0. Other boundary conditions are

$$\psi(0) = 0, \quad \psi(L) = \alpha, \quad R(L) = a.$$

a. Make dimensionless with a: $s = at$, $R = ar$, $Z = az$, $L = a\lambda$, and introduce

$$\varepsilon^2 = \frac{\sigma}{\rho g a^2}, \quad \beta = \frac{p_0}{\rho g a}. \quad \mu = \frac{\Omega^2 a}{g}.$$

Identify the dimensionless constants in terms of standard dimensionless numbers.

b. Solve the resulting problem asymptotically for $\varepsilon \to 0$. Assume that $\mu = O(1)$. Note that β and λ are unknown and therefore part of the solution.

6.2.13 Fisher's travelling wave problem

Derive an approximate solution for large c of the Fisher travelling-wave problem (Book eq. 15.19, (10.70))

$$U'' + c^2 U' + c^2 U(1 - U) = 0,$$

(a) on $(-\infty, \infty)$ with $U(-\infty) = 1$, $U(\infty) = 0$. It is no restriction to assume that $U(0) = \frac{1}{2}$.

(b) on $[0, \infty)$ and $U(0) = 0$, while the previous solution is the outer solution.

6.2.14 Nonlinear diffusion in a semi-conductor

A simple model for the nonlinear diffusion of a substitutional impurity in a certain type of semi-conductor is given by the following nonlinear generalisation of the linear diffusion equation.

$$V_\infty \frac{\partial c}{\partial t} = D_c \frac{\partial}{\partial x}\Big(v\frac{\partial c}{\partial x} - c\frac{\partial v}{\partial x}\Big)$$
$$\frac{\partial v}{\partial t} + \frac{\partial c}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2}$$

in spatial coordinate $0 < x < \infty$ and time $t > 0$. Here c denotes the concentration of impurity atoms, v the concentration of vacancies, V_∞ the equilibrium vacancy concentration, and D_v and D_c represent vacancy and impurity diffusivities respectively. The quantities V_∞, D_v and D_c are all assumed positive constants. The appropriate boundary and initial conditions are

$$\begin{array}{lll} c = C_0, & v = V_0 & \text{on } x = 0 \\ c \to 0, & v \to V_\infty & \text{as } x \to \infty \\ c = 0, & v = V_\infty & \text{at } t = 0 \end{array}$$

where C_0 and V_0 are positive constants.

Make the problem dimensionless by introducing

$$c := c/C_0, \quad v := v/V_\infty, \quad t := t/T, \quad x := x/\sqrt{D_c T}$$

where T is a timescale (does it matter which one ?). Introduce the problem parameters

$$\varepsilon^2 = D_v/D_c, \quad r = C_0/V_\infty, \quad \mu = V_0/V_\infty$$

Typically, ε is small (the literature gives an example of $\varepsilon^2 \approx 1/36$.

Derive, for small ε, a boundary layer-structured asymptotic expansion of the solution of the problem.

Tip: We saw above that the problem remains exactly the same if we scale time and space such that $t = T\tau$, $x = L\xi$ with $T = L^2$. From Buckingham's Π-theorem it then follows that the combination x^2/t is a dimensionless group. Therefore, we can conclude that c and v must be functions of the similarity variable $\eta = x/\sqrt{t}$ alone.

Before attempting to construct an approximate solution, first rewrite the set of partial differential equations into a set of ordinary differential equations (and boundary conditions) for $v = v(\eta)$ and $c = c(\eta)$ along $0 \le \eta < \infty$.

6.2.15 Heat conduction

Consider steady-state heat conduction in the rectangular region $0 \leqslant x' \leqslant L$, $-D \leqslant y' \leqslant D$. Assume that the temperature is prescribed along the edges $x' = 0$ and $x' = L$ and that the edges $y' = \pm D$ are insulated. We are interested in the problem for a slender geometry, *i.e.* $\varepsilon = D/L \ll 1$. If we normalize x with respect to L and y with respect to D, we need to solve on the rectangle $0 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ the equation

$$\varepsilon^2 T_{xx} + T_{yy} = 0, \quad T(0, y, \varepsilon) = f(y), \quad T(1, y, \varepsilon) = g(y), \quad T_y(x, \pm 1, \varepsilon) = 0$$

1. Construct an outer expansion in the form

$$T(x, y, \varepsilon) = T_0(x, y) + \varepsilon^2 T_1(x, y) + O(\varepsilon^4)$$

2. Construct appropriate inner expansions along the edges $x = 0$ and $x = 1$.
   Hint: Solve the Laplace equation on a semi-infinity strip by means of a trigonometric-exponential (Fourier) series expansion.
3. Verify that matching is possible and determine the unknown constants.
4. Solve the problem exactly and compare this with the results found.

6.2.16 Polymer extrusion

Extrusion of a polymer through a circular capillary is described by the pressure $P(t)$ in the vessel, from which the polymer is pressed, and the flow rate $Q(t)$ through the capillary.

The extrusion process is determined by the compressibility equation for the polymer in the vessel and the equation of axial momentum conservation for the flow through the capillary. In linear approximation and after neglecting the inertial effects, we obtain the following dimensionless system ($0 < \varepsilon \ll 1$)

$$\begin{aligned} \frac{\mathrm{d}P(t)}{\mathrm{d}t} &= -Q(t), && P(0) = 0; \\ \frac{\mathrm{d}Q(t)}{\mathrm{d}t} &= \frac{1}{\varepsilon}\Big(P(t) - Q(t)\Big), && Q(0) = 1. \end{aligned}$$

Determine the first term in:

1. the outer expansion;
2. the inner expansion;
3. the composite expansion.
4. Compare the composite expansion with the exact solution. Can you improve the composite expansion?

6.2.17 Torsion of a thin-walled tube

Torsion of a thin-walled tube produces a relatively large camber of the cross section. For a *clamped* tube this is blocked at the clamped cross section. This blockage mechanism plays locally (near the clamped cross section) an essential role in the distortion and stress distribution of the tube.

Consider a one-sided (at $x = 0$) clamped tube (here is x the axial co-ordinate, where $0 < x < l$.) De cross section at the other end $x = l$ is loaded by a torsional moment M, but is otherwise free. The global (*i.e.* per cross section) distortion variables are: the torsion angle $\theta(x)$ (in radians), the camber factor $\beta(x)$ (in m^{-1}) and the cross sectional shear $\kappa(x)$ (dimensionless). These three variables satisfy the following set differential equations (they follow from global equilibrium conditions for the tube) and boundary conditions (where $a_1 \dots a_4$ are constants representative of the rigidity of the cross section)

$$a_1\beta'' - a_2\beta + a_3\theta' - a_2\kappa' = 0,$$

$$a_3\beta' - a_2\theta'' + a_3\kappa'' = 0,$$

$$a_2\beta' - a_3\theta'' + a_2\kappa'' - a_4\kappa = 0,$$

and

$$\text{op } x = 0 : \quad \beta(0) = \theta(0) = \kappa(0) = 0,$$

$$\text{op } x = l \ : \quad \beta'(l) = 0, \ \ a_2\beta(l) + a_2\kappa'(l) - a_3\theta'(l) = 0, \ \ a_2\theta'(l) = M.$$

The first set boundary conditions describes the tube being fully clamped at $x = 0$, while the second tells us that the end cross section $x = l$ is free, apart from the prescribed torsional moment M.

For a rectangular tube with wall thickness t, cross sectional width and height $2b$ and $4b$, respectively, and length l (with $t \ll b \ll l$ but $(tl/b^2) = O(1)$) we have

$$a_1 = 4Eb^5t, \ \ a_2 = \frac{6E}{(1+\nu)}b^3t, \ \ a_3 = \frac{2E}{(1+\nu)}b^3t, \ \ a_4 = \frac{4Et^3}{3(1-\nu^2)b}.$$

Introduce $\varepsilon = t/b$, $0 < \varepsilon \ll 1$, and note that then also $b/l = O(\varepsilon)$.

1. Make the formulation dimensionless. Reduce the system to 1 equation for 1 unknown (for example β) plus boundary conditions.
2. Determine the first term of the outer expansion, asymptotically for $\varepsilon \to 0$.
3. The same for the inner expansion (where is the boundary layer?).
4. The same for a composite expansion.
5. Compare the composite expansion with the exact solution. Can you improve the composite expansion?

6.2.18 A visco-elastic medium forced by a piston

A linear visco-elastic medium (*Maxwell model*) is contained in a rigid cylindrical vessel, closed by a freely movable piston. As of time $t = 0$, a constant (compressive) force is applied to the piston. The vertical (*i.e.* in z–direction) displacement at a material point of the visco-elastic medium is $w = w(z,t)$. With $0 < \varepsilon \ll 1$, we have for w the following normalised, dimensionless system:

$$t > 0, \quad 0 < z < 1 : \quad \varepsilon\frac{\partial^2 w}{\partial t^2} - \frac{\partial^3 w}{\partial z^2 \partial t} - \frac{\partial^2 w}{\partial z^2} = 0,$$

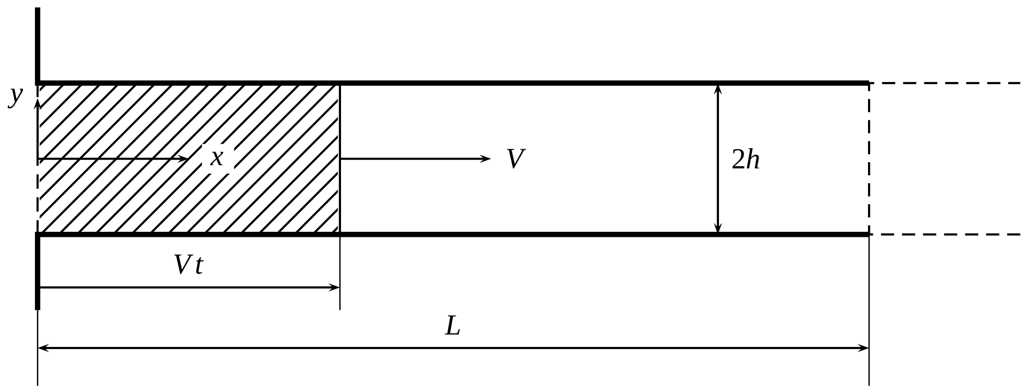
$$t = 0, \quad 0 < z < 1 : \quad w(z,0) = \frac{\partial w}{\partial t}(z,0) = 0,$$

$$t > 0, \quad z = 0 \quad : \quad w(0,t) = 0,$$

$$t > 0, \quad z = 1 \quad : \quad \frac{\partial w}{\partial z}(1,t) + \frac{\partial^2 w}{\partial z \partial t}(1,t) = -1.$$

i) Determine asymptotically for small ε (all to leading order) an outer expansion of w.
ii) Determine position and thickness of the boundary layer, and an inner expansion of w.
iii) Determine a composite expansion of w.

6.2.19 Heat conduction in fluid flow through a slit



An inviscid fluid flows with constant, uniform velocity V into a 2-dimensional slit. The slit has height $2h$ and length L, where $h/L \ll 1$ (see figure). At the entry plane of the slit, the temperature of the fluid is T_i. The upper and lower wall have temperature T_w ($T_w < T_i$). The straight front of the flow, that is at time t located at $x = Vt$, is thermally isolated from the environment.

1) Consider the temperature $T = T(x, y, t)$ for:

$$0 < x < Vt \leqslant L, \qquad -h < y < h, \qquad 0 < t < L/V.$$

Formulate the equation for T with the corresponding boundary and initial value conditions.

2) Make dimensionless.

3) Consider the ”thin-layer-approximation” (method of slow variation) for this problem, and give the solution.

4) What condition is not satisfied by the solution found under 3)? So where do you expect a boundary layer? Calculate the correction to the solution of 3) as a result of this boundary layer (accurate up to $O(h/L)$).

6.2.20 The sag of a slender plate supported at the ends

A long, slender, strip shaped plate, of width $2a$ and length $2b$ where $a/b \ll 1$, is along its long sides ($x = \pm a$) supported, while the short sides ($y = \pm b$) are free. The plate is positioned in the horizontal (x, y)–plane and is loaded by its own weight (loading per unit surface q measured in N/m^2, in the z–direction). The sag $w(x, y)$ of the plate satisfies the following differential equation

$$\left.\begin{array}{l} -a < x < a \\ -b < y < b \end{array}\right\} \qquad \nabla^4 w = \Delta\Delta w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D},$$

($\Delta = \nabla^2$, D is a plate constant, measured in Nm), plus boundary conditions

$$
\begin{aligned}
&x = \pm a \qquad w = \frac{\partial^2 w}{\partial x^2} = 0, \\
&y = \pm b \qquad \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0,
\end{aligned}
$$

(ν is Poisson's ratio, $\nu \approx 0.3$ (dimensionless)).

i) Scale the spatial coordinates like $x = a\widehat{x}$ and $y = b\widehat{y}$, and the sag like $w = (qa^4/D)\widehat{w}$, and introduce $\varepsilon = a/b \ll 1$.
ii) Determine asymptotically for small ε (to leading order) an outer expansion w_0 of w.
iii) Argue why boundary conditions can be applied at $x = \pm 1$ but not at $y = \pm 1$.
iv) Determine position and thickness of the boundary layers $y = \pm 1 + \delta(\varepsilon)\eta$, and formulate the equations for an inner expansion of w. Solve this to leading order by splitting $w = w_0 + W$ and assuming the Fourier representation $W = \sum_{n=0}^{\infty} f_n(\eta)\cos(\lambda_n x)$, where $\lambda_n = (n + \frac{1}{2})\pi$.

(Splitting off w_0 from w is advantageous because this avoids non-uniform convergence problems of the Fourier series near $x = \pm 1$.)

6.2.21 Heat conduction along cylinder walls

A circular infinitely long cylinder is applied with a thin thermally conducting outer layer, with inner radius R_1 and outer radius R_2 ($R = \frac{1}{2}(R_1 + R_2)$, $\delta = \frac{1}{2}(R_1 - R_2)$, $d/R = \delta \ll 1$). The interface between the cylinder and the layer is thermally isolated. r and θ are polar co=ordinates in a cross sectional plane of the cylinder. z is the axial co-ordinate. The cylinder rotates with constant angular velocity ω (measured in rad/sec) along the z-as.

The purpose of this layered cylinder is to transport heat from one place to another. While the outer layer is heated at one side, the same amount of heat is removed at the opposite side. This is modelled as follows:

- at the outer wall $r = R_2$ is along the part of the boundary given by $0 < \theta < \gamma$, ($\gamma \in (0, \frac{1}{2}\pi)$ a positive, constant and uniform heat flux q prescribed;
- at the outer wall $r = R_2$ is along the part of the boundary given by $\pi < \theta < \pi + \gamma$ a negative, constant and uniform heat flux $-q$ prescribed;
- along the rest of the outer wall the flux is zero.

At $t = 0$ the layer has a uniform temperature T_0.

The temperature T we are looking for is independent of z, so $T = T(r, \theta, z)$ and satisfies the following equation with initial and boundary conditions:

$$
\frac{\partial T}{\partial t} + \omega \frac{\partial T}{\partial \theta} - \kappa\Big(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\Big) = 0, \;\; R_1 < r < R_2, \;\; 0 \leqslant \theta < 2\pi, \;\; t > 0;
$$

($\kappa = k/\rho c$ is the thermal diffusion coefficient, measured in m^2/sec)

$$\begin{aligned}
T(r,\theta,0) &= T_0, && R_1 < r < R_2, \quad 0 \leqslant \theta < 2\pi;\\
\frac{\partial T}{\partial r}(R_1,\theta,t) &= \ \ 0, && 0 \leqslant \theta < 2\pi, \ \ t > 0;\\
k\frac{\partial T}{\partial r}(R_2,\theta,t) &= \ \ q, && 0 < \theta < \gamma,\\
&= -q, && \pi < \theta < \pi+\gamma,\\
&= \ \ 0, && \gamma < \theta < \pi, \quad \vee \quad \pi+\gamma < \theta < 2\pi;
\end{aligned}$$

(k measured in kg m/K $\sec^3$), plus the condition of periodicity

$$T(r,\theta+2\pi,t) = T(r,\theta,t).$$

As we are mainly interested in the temperature variation in θ, rather than in r, we introduce the mean temperature $\bar{T}$

$$\bar{T}(\theta,t) = \frac{1}{2dR}\int_{R_1}^{R_2} r\,T(r,\theta,t)\,\mathrm{d}r.$$

Show that integration in r-direction of the above system (after multiplication by r) after neglecting of $O(\delta)$-terms, yields

$$\frac{\partial \bar{T}}{\partial t} + \omega\frac{\partial \bar{T}}{\partial \theta} - \frac{\kappa}{R^2}\frac{\partial^2 \bar{T}}{\partial \theta^2} = Q(\theta), \qquad 0 \leqslant \theta < 2\pi, \ t > 0;$$

with

$$Q(\theta) = \frac{Rq}{\rho c}\,h(\theta),$$

and

$$\bar{T}(\theta,0) = T_0, \qquad \bar{T}(\theta+2\pi) = \bar{T}(\theta,t).$$

Non-dimensionalisation according to

$$\hat{t} = \omega t, \quad \hat{T} = \frac{\rho c\omega}{Rq}(T-T_0), \quad \text{en} \quad \frac{\kappa}{\omega R^2} = \varepsilon,$$

eventually leads to

$$\frac{\partial \hat{T}}{\partial t} + \frac{\partial \hat{T}}{\partial \theta} - \varepsilon\frac{\partial^2 \hat{T}}{\partial \theta^2} = h(\theta), \qquad 0 \leqslant \theta < 2\pi, \ \ t > 0,$$

and

$$\hat{T}(\theta,0) = 0, \qquad \hat{T}(\theta+2\pi,t) = \hat{T}(\theta,t),$$

with

$$\begin{aligned}
h(\theta) &= \ \ 1, && 0 < \theta < \gamma,\\
&= -1, && \pi < \theta < \pi+\gamma,\\
&= \ \ 0, && \gamma < \theta < \pi \ \ \vee \ \ \pi+\gamma < \theta < 2\pi.
\end{aligned}$$

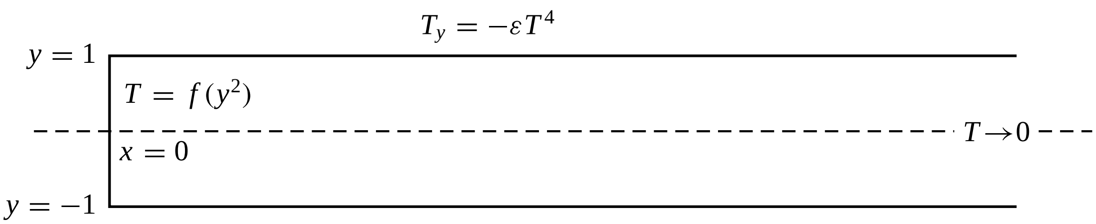
Assume in the following $0 < \varepsilon \ll 1$.

i) Determine to leading order the outer expansion of T ($\equiv \hat{T}$). Where do we observe in this solution irregularities? *Hint:* consider the gradient in θ-direction of T. So where do we expect boundary layers?

ii) Determine for one of these boundary layers (the other is similar) its thickness and the corresponding inner expansion.

6.2.22 Cooling by radiation of a heat conducting plate

Consider the stationary 2D-problem of a heat conducting semi-infinite plate, which is being heated at the short side, while along the long sides the heat disappears slowly by weak radiation (see figure).



We make lengths dimensionless by half of the thickness of the plate, and the (absolute!) temperature by a characteristic temperature at the short side.

From symmetry we consider only the upper half. In dimensionless variables the problem is then given by

$$\begin{aligned}
&0 < x,\ -1 < y < 1: && \nabla^2 T = 0\\
&x = 0, && T = f(y^2) = O(1)\\
&y = 1, && \frac{\partial T}{\partial y} = -\varepsilon T^4 \quad (0 < \varepsilon \ll 1)\\
&y = 0, && \frac{\partial T}{\partial y} = 0\\
&x \to \infty && T \to 0
\end{aligned}$$

i) As the radiation is small, the temperature decays slowly in positive x–direction. Assume that the corresponding length scale is $X = \delta(\varepsilon)x$. Determine $\delta(\varepsilon)$ by assuming a "thin layer" approximation, and balancing the radiation with the changes in x–direction.

ii) Rewrite the problem in X en y. This is a singularly perturbed problem with a boundary layer of thickness $O(\delta)$ at $x = 0$.

iii) Finish i), by determining the leading order outer solution (up to a constant). Use the fact that $T \to 0$ for $X \to \infty$.

iv) Determine the boundary layer problem, and determine the leading order boundary layer solution in the form of a Fourier expansion. (Assume for simplicity that $f(y^2) = \sum_{n=0}^{\infty} a_n \cos(n\pi y)$.) Determine by matching the unknown constant of iii).

v) What is the second order scaling function of the boundary layer solution, in other words, what is λ_1 in $T = T_0 + \lambda_1 T_1$. Write down the equation and boundary conditions for T_1. The solution is very simple.
Compare now the influx at $x = 0$ with the outflux at $y = 1$.

6.2.23 The stiffened catenary

A cable, suspended between the points $X = 0$, $Y = 0$ and $X = D$, $Y = 0$, is described as a linear elastic, geometrically non-linear inextensible bar of weight Q per unit length.

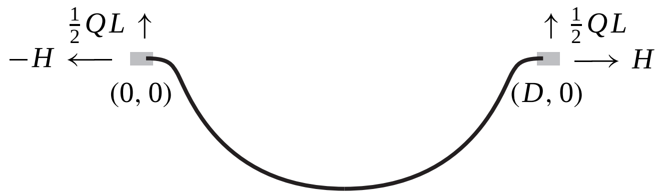


Figure 6.1: A suspended cable

At the suspension points the cable is horizontally clamped such that the cable hangs in the vertical plane through the suspension points.

The total length L of the cable is much larger than D, while the bending stiffness EI is relatively small, such that the cable is slack.

In order to keep the cable in position, the suspension points apply a reaction force, with horizontal component H resp. $-H$, and a vertical component V, resp. $QL - V$. From symmetry we already have $V = \frac{1}{2}QL$, but H is unknown.

With s the arc length along the cable, $\psi(s)$ the tangent angle with the horizon, and $X(s), Y(s)$ the cartesian co-ordinates of a point on the cable, the shape of the cable is given by

$$
\begin{aligned}
&EI\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} = H\sin\psi - (Qs - V)\cos\psi \\
&\psi(0) = \psi(L) = 0 \\
&X(L) = \int_0^L \cos\psi(s)\,\mathrm{d}s = D \\
&Y(L) = \int_0^L \sin\psi(s)\,\mathrm{d}s = 0
\end{aligned}
$$

a. Make dimensionless with L: $s = Lt$, $X = Lx$, $Y = Ly$, $D = Ld$, and introduce $\varepsilon^2 = EI/QL^3$, $h = H/QL$.

b. Solve the resulting problem asymptotically for $\varepsilon \to 0$. Assume $d = O(1)$, $h = O(1)$.

   As posed, d is known and h is unknown, and so $h = h(\varepsilon, d)$. It may be more convenient to deal with the inverse problem first, where h is known, and d results. Of course, then is $d = d(\varepsilon, h)$. Finally, after having found the relation between d and h (asymptotically), we can solve this for h and given d.

6.2.24 A boundary layer problem with x-dependent coefficients

Suppose that $y(x)$ with $0 < \varepsilon \ll 1$ satisfies the boundary value problem

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \quad y(0) = A, \quad y(1) = B,$$

while $a(x)$ and $b(x)$ are analytic in [0, 1] (*i.e.* have convergent Taylor series in any point $\in$ [0, 1].

a. If $a > 0$, find an approximate solution and show that it has a boundary layer at $x = 0$.

b. If $a < 0$, find an approximate solution and show that it has a boundary layer at $x = 1$.

c. Finally, if $a(x_0) = 0$ for $x_0 \in (0, 1)$, where $a < 0$ for $x < x_0$ and $a > 0$ for $x > x_0$, show that no boundary layer at the end points can exist, and therefore an interior layer must exist at x_0.

   Define $\beta = \dfrac{b(x_0)}{a'(x_0)}$, and show that as $x \downarrow\uparrow x_0$, the outer solutions in $x < x_0$, resp. $x > x_0$ satisfy

$$y \simeq c_{\pm}|x - x_0|^{-\beta},$$

   where the constants $c_{\pm}$ are known, but in general are not the same.

   Hence show by rescaling x and y as

$$y(x) = \left(\frac{\varepsilon}{a'(x_0)}\right)^{-\frac{1}{2}\beta} Y(X), \quad x = x_0 + \left(\frac{\varepsilon}{a'(x_0)}\right)^{\frac{1}{2}} X,$$

   the equation can be approximately written in the transition region as

$$Y'' + XY' + \beta Y = 0$$

   with matching conditions

$$Y \sim c_{\pm}|X|^{-\beta} \quad \text{as} \quad X \to \pm\infty.$$

   Solve this problem for $\beta = -1$. (Use Maple or Mathematica.)

6.2.25 A catalytic reaction problem in 1D

Consider the steady-state catalytic reaction problem of the book, section 16.8, but now in one dimension. This yields, for the concentration c, the scaled equation with boundary conditions

$$\frac{\mathrm{d}^2 c}{\mathrm{d}x^2} = \lambda \frac{c}{\alpha + c}, \qquad 0 < x < 1$$
$$c(1) = 1, \qquad c'(0) = 0.$$

Study carefully the 3D leading order asymptotic solution for $\alpha \to 0$ while $\lambda = O(1)$, given in section 16.8.3, and determine the analogous solution for 1D.

Solve the inner solution (implicitly) and find integration constants by matching. See also the note on page 546, above 16.6.5.

6.2.26 A cooling problem

Consider a synthetic fibre, shaped like an infinite cylinder of radius R, with density ρ_f, specific heat capacity c_f and thermal conductivity κ_f. At $t = 0$ the fibre has uniform temperature $T = T_0$. Inside the fibre the temperature T_f is described by

$$\rho_f c_f \frac{\partial T_f}{\partial t} - \kappa_f \nabla^2 T_f = 0.$$

Outside the fibre is air with corresponding parameters ρ_a, c_a and κ_a, and a temperature T_a given by

$$\rho_a c_a \frac{\partial T_a}{\partial t} - \kappa_a \nabla^2 T_a = 0.$$

At $t = 0$ the air temperature is equal to T_∞. For $r \to \infty$, $T_a \to T_\infty$. At the interface $r = R$, we have continuity of temperature and heat flux:

$$T_f = T_a, \quad \kappa_f \boldsymbol{n} \cdot \nabla T_f = \kappa_a \boldsymbol{n} \cdot \nabla T_a.$$

Assume that

$$\delta = \frac{\kappa_a}{\kappa_f} \ll 1, \quad \varepsilon = \frac{\kappa_f}{\rho_f c_f} \frac{\rho_a c_a}{\kappa_a} \ll 1$$

Assume cylindrical symmetry, such that $T = T(r, t)$, while

$$\nabla = \boldsymbol{e}_r \frac{\partial}{\partial r}, \quad \nabla^2 = \frac{1}{r}\Big(r \frac{\partial}{\partial r}\Big).$$

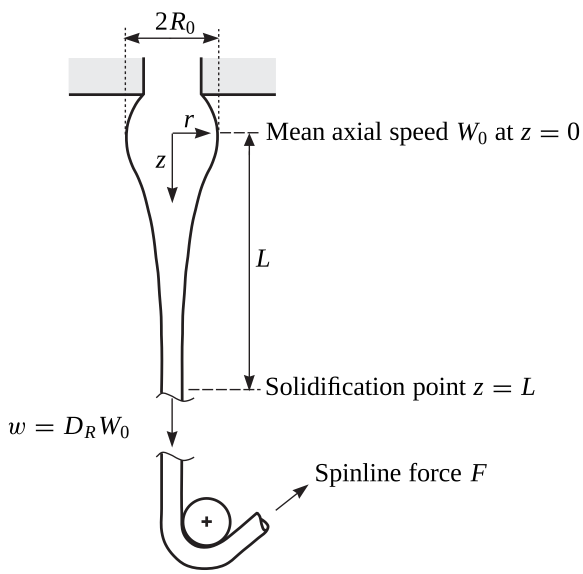
a) Scale $T = T_\infty + T_0 \theta$, $r = Rx$, and $t = (R^2 \rho_a c_a / \kappa_a)\tau$, because we are interested in the behaviour on the time scale of the heat diffusion in air.

b) Make a suitable choice to express $\delta = \delta(\varepsilon)$. Note: it is very hard to completely analyse a multi-small parameter problem asymptotically. Therefore it is *always wise* to reduce the problem to a single parameter problem by expressing one into the other.

c) Find an asymptotic approximation of θ_a and θ_f for $\varepsilon \to 0$.

6.2.27 Visco-elastic fibre spinning

The continuous stretching of viscous and elasto-viscous liquids to form fibres is a primary manufacturing process for textiles and glass fibres. The melt spinning process for the manufacture of fibres is shown schematically in the figure. Molten material is extruded through a small hole into cross-flowing ambient air at a temperature below the solidification temperature of the material. The solidified polymer or glass is wound up on a reel moving at a higher speed than the mean extrusion velocity, resulting in thinning of the filament.



Conservation of momentum for a Maxwell model, ignoring inertia, surface tension, air friction and gravity, yields

$$\tau_{zz} + \lambda \left[w \frac{\mathrm{d}\tau_{zz}}{\mathrm{d}z} - 2\tau_{zz} \frac{\mathrm{d}w}{\mathrm{d}z} \right] = 2\eta \frac{\mathrm{d}w}{\mathrm{d}z},$$
$$\tau_{rr} + \lambda \left[w \frac{\mathrm{d}\tau_{rr}}{\mathrm{d}z} + \tau_{rr} \frac{\mathrm{d}w}{\mathrm{d}z} \right] = -\eta \frac{\mathrm{d}w}{\mathrm{d}z}$$

where w is the cross-wise averaged axial velocity, η denotes the coefficient of dynamic viscosity, λ denotes the coefficient of elastic relaxation, and $w(0) = W_0$ is the initial axial velocity. Note that in practice W_0 is not given but a result from the drawing force F applied at an end position $z = L$ (the solidification point).

We may assume that, due to surface tension, the cross-sectional shape of the fibre is circular, of radius (say) $R = R(z)$. We define $R_0 = R(0)$.

The normal components of the stress tensor $\boldsymbol{n} \cdot \sigma$ vanish at the surface $r = R(z)$, leading to

$$\sigma_{rz} - R_z \sigma_{zz} = 0, \quad \sigma_{rr} - R_z \sigma_{rz} = 0 \quad \text{at } r = R.$$

For small R_z (the assumption of a slowly varying diameter) it follows that $\sigma_{rr} = (R_z)^2 \sigma_{zz} \simeq 0$. Furthermore, when we integrate along a cross section

$$\int_0^{2\pi} \int_0^R \left[\frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \Big(r \tau_{rr} \Big) \right] r \,\mathrm{d}r \mathrm{d}\theta = 2\pi \frac{\mathrm{d}}{\mathrm{d}z} \int_0^R \sigma_{zz} r \,\mathrm{d}r + 2\pi R_z \left[-r\sigma_{zz} + r\sigma_{rz} \right]_{r=R} =$$
$$2\pi \frac{\mathrm{d}}{\mathrm{d}z} \int_0^R \sigma_{zz} r \,\mathrm{d}r = 0$$

For σ_{zz} practically constant along a cross section this leads to

$$\pi R^2 \sigma_{zz} = F.$$

To make the problem tractable we ignore possible entrance effect and assume $\tau_{rr} = 0$ at $z = 0$. Altogether we have

$$\sigma_{zz} = -p + \tau_{zz} = \frac{F}{\pi R^2},$$
$$\sigma_{rr} = -p + \tau_{rr} = 0.$$

Conservation of mass of incompressible flow yields the axial volume flux

$$Q = \pi R^2 w = \pi R_0^2 W_0.$$

We make dimensionless

$$w = W_0 v, \quad z = Ly, \quad \tau_{zz} = T\frac{F}{\pi R_0^2}, \quad \tau_{rr} = P\frac{F}{\pi R_0^2}$$

and introduce the dimensionless parameters

$$q = \frac{\eta \pi R_0^2 W_0}{FL}, \quad \varepsilon = \frac{\lambda W_0}{L}$$

where $q = O(1)$ is the ratio between viscous and axial forces and ε is called the Weissenberg number. We will analyse asymptotically the problem of small ε.

Noting that $R_0^2/R^2 = w/W_0 = v$, we have

$$T - P = v,$$
$$T + \varepsilon\left[v\frac{\mathrm{d}T}{\mathrm{d}y} - 2T\frac{\mathrm{d}v}{\mathrm{d}y}\right] = 2q\frac{\mathrm{d}v}{\mathrm{d}y},$$
$$P + \varepsilon\left[v\frac{\mathrm{d}P}{\mathrm{d}y} + P\frac{\mathrm{d}v}{\mathrm{d}y}\right] = -q\frac{\mathrm{d}v}{\mathrm{d}y}$$

with

$$v(0) = 1 \text{ and } P(0) = 0.$$

a) Eliminate P and $\frac{\mathrm{d}}{\mathrm{d}y}T$ to find

$$v + \varepsilon v'(2v - 3T) = 3qv'$$

where $v' = \frac{\mathrm{d}}{\mathrm{d}y}v$.

b) Eliminate T to find the single equation for v

$$\varepsilon\big(v^2 v'' - v(v')^2\big) + 2\varepsilon^2 v(v')^3 - vv' + 3q(v')^2 = 0$$

Note that this is a 2d-order autonomous ordinary differential equation.

c) Introduce $\phi(x) = v'(y)$ and $x = \ln(v)$ and reformulate the equation for v into a 1st-order non-autonomous ordinary differential equation with boundary conditions, for $\phi = \phi(x, \varepsilon)$.

d) Solve this equation by matched asymptotic expansions for small ε up to and including terms of $O(\varepsilon)$. It is more work, but it is possible to get also the $O(\varepsilon^2)$-terms. Note that a boundary layer at $x = 0$ may be anticipated.

e) Rewrite the leading order outer solution into a solution for v. This is the solution for a purely viscous fibre, with negligible elastic effects.

6.2.28 The weather balloon

In this exercise we will consider a simple model of the dynamics of a weather balloon in a non-uniform but stationary atmosphere. In order to make analysis possible, we consider the Standard Atmosphere which is explicitly given. We start with some preliminary information.

Ideal Gas

In the ideal gas model the relation between pressure p, density ρ and absolute temperature T is approximated by

$$p = \rho R T$$

where R is constant, the so-called gas constant.

From known reference values at a pressure of $p = 101.325$ kPa we can derive the respective gas constants for air, helium and hydrogen as follows

Air :	$T = 288.15\,\mathrm{K},\ \rho = 1.22500$ g/l $\rightarrow R_a = p/\rho T = 287.053$
Helium:	$T = 273.15\,\mathrm{K},\ \rho = 0.1786$ g/l $\rightarrow R_h = p/\rho T = 2076.99$
Hydrogen:	$T = 273.15\,\mathrm{K},\ \rho = 0.08988$ g/l $\rightarrow R_w = p/\rho T = 4127.17$

Table 6.1: Gas constants

Standard Atmosphere

A simplified model of the atmosphere, know as the Standard Atmosphere, is useful as reference. It starts with a simple relation between temperature T and height h (above sea level), the assumption of hydrostatic equilibrium $\mathrm{d}p/\mathrm{d}h = -g\rho$ and the ideal gas law.

In the troposphere ($0 \leqslant h \leqslant 11$ km) the air temperature T is assumed to decay linearly, such that pressure and density at height h above sea level are given by

$$\begin{aligned}
T &= T_0\left(1 - \frac{h}{L}\right)\ \mathrm{K} && \text{with } T_0 = 288.15\ \mathrm{K}, \quad L = 10^3 T_0/6.5 = 44331\ \mathrm{m}\\
p &= p_0\left(\frac{T}{T_0}\right)^{\alpha}\ \mathrm{Pa}, && \text{with } p_0 = 101325\ \mathrm{Pa}, \quad g = 9.81\ \mathrm{m/s^2}, \quad \alpha = \frac{gL}{T_0 R} = 5.256\\
\rho &= \rho_0\left(\frac{T}{T_0}\right)^{\alpha-1}\ \mathrm{kg/m^3}, && \text{with } \rho_0 = 1.225\ \mathrm{kg/m^3}.
\end{aligned}$$

The Balloon

A balloon of mass m and negligible own weight is filled with a gas of density ρ_g (helium or hydrogen). The volume of the balloon is $V = m/\rho_g$. We assume that the balloon is made of arbitrarily flexible and extensible material such that it always attains a spherical shape. If the radius is r, the volume is $V = \frac{4}{3}\pi r^3$ and the cross sectional surface $A = \pi r^2$.

The weight of the balloon is gm, while the Archimedean upward force is $g\rho_a V$. The balloon is released at time $t = 0$ from stationary state at sea level $h = 0$ in a steady atmosphere, describable by the Standard Atmosphere. We are interested in its height h and velocity $\dot{h}$ at time t.

As the balloon surface is perfectly flexible, it does not add extra pressure, and the pressure inside and outside the balloon are equal: $p_a = p_g$. (In reality, no balloon is of course perfectly flexible, and at some point the surface will not expand further. This is where the balloon stops rising.)

The temperature inside the balloon will be the temperature outside with some delay, to be modelled by a temperature diffusion model which we will not include here. One extreme is to model $T_g = T_0$ for very fast balloons and the other extreme is to model $T_g = T_a$ for very slow balloons. For mathematical convenience we will consider the second assumption, but the other case is similar.

The Forces

The balloon is subject to inertia $-m\,\ddot{h}$, buoyancy force $g\rho_a V - gm$ and air drag $-\frac{1}{2}\rho_a\,\dot{h}^2\,C_d A$, where drag coefficient C_d is for a sphere (of not too low and not too high Reynolds number) in the order of 0.5 [Batchelor 1967, p. 341].

Together these forces cancel out each other, so altogether we have the following equation for the dynamics of the balloon

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = g\rho_a V - gm - \frac{1}{2}\rho_a\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 C_d A.$$

Simplifying the problem to obtain a model

By using the above relations we find

$$\begin{aligned}
m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} &= gm\frac{\rho_a}{\rho_g} - gm - \frac{1}{2}\pi C_d\rho_a\left(\frac{3V}{4\pi}\right)^{2/3}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\\
\frac{\mathrm{d}^2h}{\mathrm{d}t^2} &= g\left(\frac{\rho_a}{\rho_g} - 1\right) - \frac{\frac{1}{2}\pi C_d}{m}\rho_a\left(\frac{3m}{4\pi\rho_g}\right)^{2/3}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\\
\frac{\mathrm{d}^2h}{\mathrm{d}t^2} &= g\left(\frac{\rho_a}{\rho_g} - 1\right) - \frac{\frac{1}{2}\pi C_d}{m^{1/3}}\left(\frac{3}{4\pi}\right)^{2/3}\frac{\rho_a}{\rho_g^{2/3}}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2
\end{aligned}$$

Now we use the equal pressures inside and outside

$$1 = \frac{p_a}{p_g} = \frac{\rho_a R_a T_a}{\rho_g R_g T_g} = \frac{\rho_a R_a}{\rho_g R_g}$$

such that

$$\rho_g = \frac{R_a}{R_g}\rho_a.$$

If we further introduce $q = \frac{1}{6}(\alpha - 1) = 0.7093$, we have

$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = g\left(\frac{R_g}{R_a} - 1\right) - \frac{1}{2}\pi C_d\left(\frac{3R_g}{4\pi R_a}\right)^{2/3}\left(\frac{\rho_a}{m}\right)^{1/3}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2,$$

and eventually an equation in h

$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = g\left(\frac{R_g}{R_a}-1\right) - \frac{1}{2}\pi C_d \left(\frac{3R_g}{4\pi R_a}\right)^{2/3}\left(\frac{\rho_0}{m}\right)^{1/3}\left(1-\frac{h}{L}\right)^{2q}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2.$$

Non-dimensionalisation

We make dimensionless by assuming

$$h = \ell H, \quad t = \tau s$$

such that

$$\frac{\ell H''}{\tau^2} = g\left(\frac{R_g}{R_a}-1\right) - \frac{1}{2}\pi C_d \left(\frac{3R_g}{4\pi R_a}\right)^{2/3}\left(\frac{\rho_0}{m}\right)^{1/3}\left(1-\frac{\ell}{L}H\right)^{2q}\left(\frac{\ell H'}{\tau}\right)^2.$$

A possible lengthscale could be a radius r, for example at $t = 0$, but this is obviously for the global dynamics only relevant in a very indirect way. The most natural, inherent, length scale for h is obviously $\ell = L$. A reference time scale is less obvious. If the buoyancy and drag are dominating for most of the time, we choose τ such that these forces balance:

$$\tau = \sqrt{\frac{1}{2}\pi C_d \frac{R_a}{R_g - R_a}\left(\frac{3R_g}{4\pi R_a}\right)^{2/3}\frac{L^2}{g}\left(\frac{\rho_0}{m}\right)^{1/3}}.$$

For 1 kg helium this characteristic time is $\tau = 6235.47$ s or 104 minutes. For 1 kg hydrogen, it is $\tau = 5352.07$ s or 89 minutes.

The relative importance of inertia against the other forces is characterised by the parameter

$$\varepsilon = \frac{\frac{L}{\tau^2}}{g\left(\frac{R_g}{R_a}-1\right)} = \frac{2}{\pi C_d}\left(\frac{4\pi R_a}{3R_g}\right)^{2/3}\frac{1}{L}\left(\frac{m}{\rho_0}\right)^{1/3},$$

which amounts here to $\varepsilon = 1.86\cdot 10^{-5}$ for 1 kg helium and $\varepsilon = 1.18\cdot 10^{-5}$ for 1 kg hydrogen. So ε is a small parameter and most of the time inertia is unimportant. Hence, τ is the typical time it takes for h to vary by an amount comparable with L.

We finally obtain the model in its most transparent form

$$\varepsilon H'' = 1 - (1-H)^{2q}(H')^2$$

with initial conditions $H(0) = H'(0) = 0$. Because of these we may assume that $H'(s) \geqslant 0$.

This dimensionless form of the problem confirms that the inertial forces $\varepsilon H''$ are negligible during most of the balloon's flight. Neglecting this term altogether, on the other hand, reduces the order of the differential equation from two to one, which is not possible since we have two initial conditions. The answer is of course that there exists a boundary layer near the starting time $s = 0$. We will solve this problem therefore by application of the Method of Matched Asymptotic Expansions.

Asymptotic analysis

Solve this problem asymptotically for small ε up to first and second order.

6.2.29 A chemical reaction-diffusion problem (singular limit)

Reconsider the chemical reaction-diffusion problem of problem 3.4.8 (page 55)

$$\begin{aligned} &\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\frac{\mathrm{d}c}{\mathrm{d}r}\Big) = \phi^2 c^n, \quad 0 < r < 1, \\ &c(1) = 1, \;\; c'(0) = 0, \end{aligned}$$

but now for the asymptotic behaviour of c when $\varepsilon = \phi^{-1} \to 0$. Solve first the exact solution for $n = 1$ to guess the general structure. Find the leading order inner and outer solution.
Hint. Introduce $y = rc$.

6.2.30 An internal boundary layer (Oxford, OCIAM, 2003)

The function $y(x, \varepsilon)$ satisfies the equation

$$\varepsilon y'' + yy' - y = 0 \qquad \text{for} \quad x \in (0, 1)$$

Consider the following singularly perturbed boundary value problems for $\varepsilon \to 0$.

a) Suppose that $y(0) = 0$ and $y(1) = 3$. Show that a solution can be found with a boundary layer at $x = 0$. Give leading order approximations for the inner and outer expansions.
b) Suppose that $y(0) = -\frac{3}{4}$ and $y(1) = \frac{5}{4}$. Show that a solution is possible having an interior layer where y jumps from $-M$ to M, for some M. Find the leading order matched asymptotic expansions.

6.2.31 The Van der Pol equation with strong damping

By a suitable coordinate transformation we can write Van der Pol's equation with strong damping as

$$\varepsilon y'' + y - (1 - y^2)y' = 0 \qquad \varepsilon \to 0.$$

We are interested in a *periodic* solution, so the choice of the initial conditions is part of the problem. By seeking appropriate "outer" and "transition" layers, show that an approximate periodic solution can be constructed to leading order in the form of matched asymptotic expansions. Show that the period is approximately $3 - 2\ln 2$.

6.2.32 A beam under tension resting on an elastic foundation

An elastic beam of bending stiffness EI is resting on an elastic foundation of modulus $k(s)$, while it is under tension T and under a distributed downward force per length $p(s)$. The distance along the beam is s. The small vertical deflection w of the beam satisfies the ordinary differential equation

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}s^4} - T\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} + kw = p$$

Assume that there is an inherent length scale L, such that we can scale $s = Lx$. Assume that k is typically of order K, *i.e.* $k(s) = K\kappa(x)$, p is typically of order P, *i.e.* $p(s) = Pf(x)$, and w is typically of order W, *i.e.* $w(s) = Wy(x)$.

a) Show, by choosing suitable lengths L and W that the above equation can be written in dimensionless form as

$$\varepsilon^2(y'''' + \kappa y) - y'' = f.$$

b) Suppose that the beam rests on a foundation with modulus that varies linearly along the length of the beam, *i.e.* $\kappa(x) = 1 + mx$. Other than the tension, there is no external forcing, *i.e.* $f(x) = 0$. Model the beam as semi-infinite along $x \in [0, \infty)$, with a horizontally clamped, prescribed deflection at $x = 0$ and (due to the increasing foundation modulus) no deflection for $x \to \infty$. We arrive then at the differential equation and boundary conditions

$$\varepsilon^2(y'''' + (1 + mx)y) - y'' = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y(x), y'(x) \to 0 \quad (x \to \infty).$$

If $m = O(1)$ and $\varepsilon \to 0$, find a first-order asymptotic approximation of y based on a boundary layer structure near $x = 0$.

HINT. Don't be fooled by the form of the equation. The outer variable is *not* necessarily x, and should be found from a judicious balancing of terms of the equation.

Chapter 7

Multiple Scales, WKB and Resonance

7.1 Theory

7.1.1 Multiple Scales: general procedure

Suppose a function $\varphi(x, \varepsilon)$ depends on more than one length scale acting together, for example x, εx, and $\varepsilon^2 x$. Then the function does not have a regular expansion on the full domain of interest, $x \leqslant O(\varepsilon^{-2})$ say. It is not possible to bring these different length scales together by a simple coordinate transformation, like in the method of slow variation or the Lindstedt-Poincaré method, or to split up our domain in subdomains like in the method of matched asymptotic expansions. Therefore we have to find another way to construct asymptotic expansions, valid in the full domain of interest. The approach that is followed in the *method of multiple scales* is at first sight rather radical: the various length scales are temporarily considered as independent variables: $x_1 = x$, $x_2 = \varepsilon x$, $x_3 = \varepsilon^2 x$, and the original function φ is identified with a more general function $\psi(x_1, x_2, x_3, \varepsilon)$ depending on a higher dimensional independent variable.

$$\varphi(x, \varepsilon) = A(\varepsilon)\, \mathrm{e}^{-\varepsilon x} \cos(x - \theta(\varepsilon)) \;\text{ becomes }\; \psi(x_1, x_2, \varepsilon) = A(\varepsilon)\, \mathrm{e}^{-x_2} \cos(x_1 - \theta(\varepsilon)). \qquad \square$$

Since this identification is not unique, we may add constraints such that this auxiliary function ψ does have a Poincaré expansion on the full domain of interest. After having constructed this expansion, it may be associated to the original function along the line $x_1 = x$, $x_2 = \varepsilon x$, $x_3 = \varepsilon^2 x$.

The technique, utilizing this difference between small scale and large scale behaviour is the method of multiple scales. As with most approximation methods, this method has grown out of practice, and works well for certain types of problems. Typically, the multiple scale method is applicable to problems with on the one hand a certain global quantity (energy, power), which is conserved or almost conserved, controlling the amplitude, and on the other hand two rapidly interacting quantities (kinetic and potential energy), controlling the phase. Usually, this describes slowly varying waves, affected by small effects during a long time. Intuitively, it is clear that over a short distance (a few wave lengths) the wave only sees constant conditions and will propagate approximately as in the constant case, but over larger distances it will somehow have to change its shape in accordance with its new environment.

7.1.2 A practical example: a damped oscillator

We will illustrate the method by considering a damped harmonic oscillator

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\varepsilon\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \quad (t \geqslant 0), \qquad y(0) = 0,\ \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 1 \tag{7.1}$$

with $0 < \varepsilon \ll 1$. The exact solution is readily found to be

$$y(t) = \mathrm{e}^{-\varepsilon t}\,\frac{\sin\bigl(\sqrt{1-\varepsilon^2}\,t\bigr)}{\sqrt{1-\varepsilon^2}}. \tag{7.2}$$

A naive approximation of this $y(t)$, for small ε and fixed t, would give

$$y(t) = \sin t - \varepsilon t\sin t + O(\varepsilon^2), \tag{7.3}$$

which appears to be useful for $t = O(1)$ only. For large t the approximation becomes incorrect:

1) if $t \geqslant O(\varepsilon^{-1})$ the second term is of equal importance, or larger, as the first term and nothing is left over of the slow exponential decay;
2) if $t \geqslant O(\varepsilon^{-2})$ the phase has an error of $O(1)$, or larger, giving an approximation of which even the sign may be in error.

We would obtain a far better approximation if we adopted two different time variables, *viz.* $T = \varepsilon t$ and $\tau = \sqrt{1-\varepsilon^2}\,t$, and changed to $y(t,\varepsilon) = Y(\tau, T, \varepsilon)$ where

$$Y(\tau, T, \varepsilon) = \mathrm{e}^{-T}\,\frac{\sin(\tau)}{\sqrt{1-\varepsilon^2}}.$$

It is easily verified that a Taylor series of Y in ε yields a regular expansion for all t.

If we construct a straightforward approximate solution directly from equation (7.1), we would get the same approximation as in (7.3), which is too limited for most applications. However, knowing the character of the error, we may try to avoid them and look for the auxiliary function Y, instead of y. As we, in general, do not know the occurring time scales, their determination becomes part of the problem.

Suppose we can expand

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots . \tag{7.4}$$

Substituting in (7.1) and collecting equal powers of ε gives

$$O(\varepsilon^0): \quad \frac{\mathrm{d}^2y_0}{\mathrm{d}t^2} + y_0 = 0 \qquad \text{with } y_0(0) = 0,\ \frac{\mathrm{d}y_0(0)}{\mathrm{d}t} = 1,$$

$$O(\varepsilon^1): \quad \frac{\mathrm{d}^2y_1}{\mathrm{d}t^2} + y_1 = -2\frac{\mathrm{d}y_0}{\mathrm{d}t} \quad \text{with } y_1(0) = 0,\ \frac{\mathrm{d}y_1(0)}{\mathrm{d}t} = 0.$$

We then find

$$y_0(t) = \sin t, \qquad y_1(t) = -t\sin t, \qquad \textit{etc.}$$

which reproduces indeed expansion (7.3). The straightforward, Poincaré type, expansion (7.4) breaks down for large t, when $\varepsilon t \geqslant O(1)$. It is important to note that this caused by the fact that any y_n is excited in its eigenfrequency (by the "source"-terms $-2\mathrm{d}y_{n-1}/\mathrm{d}t$), resulting in resonance. We recognise the generated algebraically growing terms of the type $t^n \sin t$ and $t^n \cos t$, called *secular terms* (definition 5.1.1). Apart from being of limited validity, the expansion reveals nothing of the real structure of the solution, and we change our strategy to looking for an auxiliary function dependent on different time scales. We start with the hypothesis that, next to a fast time scale t, we have the slow time scale

$$T = \varepsilon t. \tag{7.5}$$

Then we identify the solution y with a suitably chosen other function Y that depends on both variables t and T

$$Y(t, T, \varepsilon) = y(t, \varepsilon).$$

There exist infinitely many functions $Y(t, T, \varepsilon)$ that are equal to $y(t, \varepsilon)$ along the line $T = \varepsilon t$ in (t, T)-space. So we have now some freedom to prescribe additional conditions. With the unwelcome appearance of secular terms in mind it is natural to think of conditions, to be chosen such that no secular terms occur when we construct an approximation.

Since the time derivatives of y turn into partial derivatives of Y, *i.e.*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial Y}{\partial t} + \varepsilon \frac{\partial Y}{\partial T},$$

equation (7.1) becomes for Y

$$\frac{\partial^2 Y}{\partial t^2} + Y + 2\varepsilon\Big(\frac{\partial Y}{\partial t} + \frac{\partial^2 Y}{\partial t \partial T}\Big) + \varepsilon^2\Big(\frac{\partial^2 Y}{\partial T^2} + 2\frac{\partial Y}{\partial T}\Big) = 0. \tag{7.6}$$

Assume the expansion

$$Y(t, T, \varepsilon) = Y_0(t, T) + \varepsilon Y_1(t, T) + \varepsilon^2 Y_2(t, T) + \cdots \tag{7.7}$$

and substitute this into (7.6) to obtain to leading orders

$$\begin{aligned} &\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \\ &\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial Y_0}{\partial t} - 2\frac{\partial^2 Y_0}{\partial t \partial T}, \end{aligned}$$

with initial conditions

$$\begin{aligned} &Y_0(0, 0) = 0, \qquad \frac{\partial}{\partial t} Y_0(0, 0) = 1, \\ &Y_1(0, 0) = 0, \qquad \frac{\partial}{\partial t} Y_1(0, 0) = -\frac{\partial}{\partial T} Y_0(0, 0). \end{aligned}$$

The solution for Y_0 is easily found to be

$$Y_0(t, T) = A_0(T)\sin(t - \theta_0(T)) \qquad \text{with} \quad A_0(0) = 1, \theta_0(0) = 0.$$

This gives a right-hand side for the Y_1-equation of

$$-2\Big(A_0 + \frac{\partial A_0}{\partial T}\Big)\cos(t - \theta_0) + 2A_0\frac{\partial \theta_0}{\partial T}\sin(t - \theta_0).$$

No secular terms occur (no resonance between Y_1 and Y_0) if these terms vanish:

$$A_0 + \frac{\partial A_0}{\partial T} = 0 \quad \text{yielding} \quad A_0 = \mathrm{e}^{-T}, \qquad \frac{\partial \theta_0}{\partial T} = 0 \quad \text{yielding} \quad \theta_0 = 0.$$

Together we have indeed constructed an approximation of (7.2), valid for $t \le O(\varepsilon^{-1})$.

$$y(t, \varepsilon) = \mathrm{e}^{-\varepsilon t} \sin t + O(\varepsilon).$$

Note (this is typical of this approach), that we determined Y_0 only on the level of Y_1, but without having to solve Y_1 itself.

The present approach is by and large the multiple scale technique in its simplest form. Variations on this theme are sometimes necessary. For example, we have not completely got rid of secular terms. On a longer time scale ($t = O(\varepsilon^{-2})$) we have again resonance in Y_2 because of the “source” $\mathrm{e}^{-T} \sin t$, yielding terms $O(\varepsilon^2 t)$. We see that a second time scale $T_2 = \varepsilon^2 t$ is necessary. From the exact solution we may infer that these longer time scales are not really independent and it may be worthwhile to try a fast time of strained coordinates type: $\tau = \omega(\varepsilon)t = (1+\varepsilon^2\omega_1+\varepsilon^4\omega_4+\ldots)t$. In the present example we would recover $\omega(\varepsilon) = \sqrt{1-\varepsilon^2}$.

7.1.3 The air-damped resonator.

In dimensionless form this is given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \varepsilon \frac{\mathrm{d}y}{\mathrm{d}t}\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right| + y = 0, \qquad \text{with} \quad y(0) = 1,\ \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 0. \tag{7.8}$$

By rewriting the equation into the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big[\tfrac{1}{2}(y')^2 + \tfrac{1}{2}y^2\big] = -\varepsilon (y')^2 |y'|$$

and assuming that y and $y' = O(1)$, it may be inferred that the damping acts on a time scale of $O(\varepsilon^{-1})$. So we conjecture the presence of the slow time variable $T = \varepsilon t$ and introduce a new dependent variable Y that depends on both t and T. We have

$$T = \varepsilon t, \qquad y(t, \varepsilon) = Y(t, T, \varepsilon), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial Y}{\partial t} + \varepsilon \frac{\partial Y}{\partial T},$$

and obtain for equation (7.8)

$$\frac{\partial^2 Y}{\partial t^2} + Y + \varepsilon \left(2\frac{\partial^2 Y}{\partial t \partial T} + \frac{\partial Y}{\partial t}\left|\frac{\partial Y}{\partial t}\right| \right) + O(\varepsilon^2) = 0$$
$$Y(0,0,\varepsilon) = 1, \quad \Big(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}\Big) Y(0,0,\varepsilon) = 0.$$

The error of $O(\varepsilon^2)$ results from the approximation $\frac{\partial}{\partial t}Y + \varepsilon\frac{\partial}{\partial T}Y = \frac{\partial}{\partial t}Y + O(\varepsilon)$, and is of course only valid outside a small neighbourhood of the points where $\frac{\partial}{\partial t}Y = 0$. We expand

$$Y(t, T, \varepsilon) = Y_0(t, T) + \varepsilon Y_1(t, T) + O(\varepsilon^2),$$

to find for the leading order

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \quad \text{with} \quad Y_0(0,0) = 1,\ \frac{\partial}{\partial t} Y_0(0,0) = 0.$$

The solution is given by

$$Y_0 = A_0(T)\cos(t - \theta_0(T)), \quad \text{where} \quad A_0(0) = 1,\ \theta_0(0) = 0.$$

For the first order we have the equation

$$\begin{aligned}\frac{\partial^2 Y_1}{\partial t^2} + Y_1 &= -2\frac{\partial^2 Y_0}{\partial t \partial T} - \frac{\partial Y_0}{\partial t}\left|\frac{\partial Y_0}{\partial t}\right| \\ &= 2\frac{\mathrm{d}A_0}{\mathrm{d}T}\sin(t-\theta_0) - 2A_0\frac{\mathrm{d}\theta_0}{\mathrm{d}T}\cos(t-\theta_0) + A_0^2 \sin(t-\theta_0)|\sin(t-\theta_0)|,\end{aligned}$$

with corresponding initial conditions. The secular terms are suppressed if the first harmonics of the right-hand side cancel. For this we use the Fourier series expansion

$$\sin(t)\,|\sin(t)| = -\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)t}{(2n-1)(2n+1)(2n+3)}.$$

We obtain the equations

$$2\frac{\mathrm{d}A_0}{\mathrm{d}T} + \frac{8}{3\pi}A_0^2 = 0 \quad \text{and} \quad \frac{\mathrm{d}\theta_0}{\mathrm{d}T} = 0,$$

with solution $\theta_0(T) = 0$ and

$$A_0(T) = \frac{1}{1 + \frac{4}{3\pi}T}.$$

Altogether we have the approximate solution

$$y(t,\varepsilon) = \frac{\cos(t)}{1 + \frac{4}{3\pi}\varepsilon t} + O(\varepsilon).$$

This approximation appears to be remarkably accurate. See Figure 7.1 where plots, made for a parameter value of $\varepsilon = 0.1$, of the approximate and a numerically "exact" solution are hardly distinguishable. A maximum difference is found of 0.03.

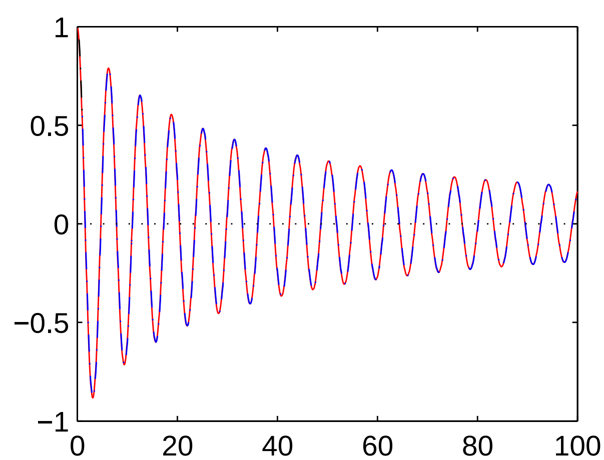


Figure 7.1: Plots of the approximate and a numerically "exact" solution $y(t,\varepsilon)$ of the air-damped resonator problem for $\varepsilon = 0.1$.

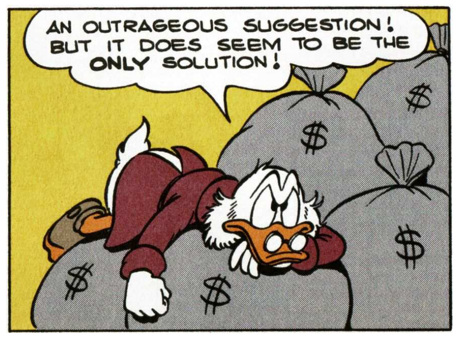
7.1.4 The WKB Method: slowly varying fast time scale

The method of multiple scales fails when the slow variation is caused by external effects, like a slowly varying problem parameter. In this case the nature of the slow variation is not the same for all time, but may vary. This is demonstrated by the following example. Consider the problem

$$\ddot{x} + \kappa(\varepsilon t)^2 x = 0, \quad x(0,\varepsilon) = 1, \quad \dot{x}(0,\varepsilon) = 0,$$

where $\kappa = O(1)$. It seems plausible to assume 2 time scales: a fast one $O(\kappa^{-1}) = O(1)$ and a slow one $O(\varepsilon^{-1})$. So we introduce next to t the slow scale $T = \varepsilon t$, and rewrite $x(t,\varepsilon) = X(t, T, \varepsilon)$. We expand $X = X_0 + \varepsilon X_1 + \ldots$, and obtain $X_0 = A_0(T)\cos(\kappa(T)t - \theta_0(T))$. Suppressing secular terms in the equation for X_1 requires $A_0' = \kappa' t - \theta_0' = 0$, which is impossible.

Here, the fast time scale is slowly varying itself and the fast variable is to be strained locally by a suitable strain function. This sounds far-fetched, but is in fact quite simple: we introduce a fast time scale via a slowly varying function.



Often, it is convenient to write this function in the form of an integral, because it always appears in the equations after differentiation. For a function ω to be found

$$\tau = \int_0^t \omega(\varepsilon t', \varepsilon)\, \mathrm{d}t' = \frac{1}{\varepsilon}\int_0^T \omega(z,\varepsilon)\, \mathrm{d}z, \quad \text{where } T = \varepsilon t,$$

while for $x(t,\varepsilon) = X(\tau, T, \varepsilon)$ we have

$$\dot{x} = \omega X_\tau + \varepsilon X_T \qquad \text{and} \qquad \ddot{x} = \omega^2 X_{\tau\tau} + \varepsilon\omega_T X_\tau + 2\varepsilon\omega X_{\tau T} + \varepsilon^2 X_{TT}.$$

After expanding $X = X_0 + \varepsilon X_1 + \ldots$ and $\omega = \omega_0 + \varepsilon\omega_1 + \ldots$ we obtain

$$\begin{aligned} &\omega_0^2 X_{0\tau\tau} + \kappa^2 X_0 = 0, \\ &\omega_0^2 X_{1\tau\tau} + \kappa^2 X_1 = -2\omega_0\omega_1 X_{0\tau\tau} - \omega_{0T} X_{0\tau} - 2\omega_0 X_{0\tau T}. \end{aligned} \tag{7.9}$$

The leading order solution is $X_0 = A_0(T)\cos(\lambda(T)\tau - \theta_0(T))$, where $\lambda = \kappa/\omega_0$. The right-hand side of (7.9) is then

$$2\omega_0 A_0 \lambda(\omega_1\lambda + \lambda_T\tau - \theta_{0T})\cos(\lambda\tau - \theta_0) + (A_0\lambda)^{-1}(\omega_0 A_0^2\lambda^2)_T \sin(\lambda\tau - \theta_0).$$

Suppression of secular terms requires $\lambda_T = 0$. Without loss of generality we can take $\lambda = 1$, or $\omega_0 = \kappa$. Then we need $\omega_1 = \theta_{0T}$, which just yields that $\lambda\tau - \theta_0 = \tau - \theta_0 = \varepsilon^{-1}\int^T \omega(z)\,\mathrm{d}z - \int^T \omega_1(z)\,\mathrm{d}z = \varepsilon^{-1}\int^T \omega_0(z)\,\mathrm{d}z + O(\varepsilon)$. In other words, we may just as well take $\omega_1 = 0$ and $\theta_0 = \gamma$ a constant. Finally we have $\omega_0 A_0^2 \lambda^2 = \kappa A_0^2 = a$ a constant[1], or $A_0 = a/\sqrt{\kappa}$. Altogether we have

$$x(t) \simeq \frac{a}{\sqrt{\kappa(\varepsilon t)}}\cos\Big(\int_0^t \kappa(\varepsilon t')\,\mathrm{d}t' - \gamma\Big)$$

[1] A conserved slow-time quantity like κA_0^2 is called an *adiabatic invariant*.

The introduction of a slow time scale together with the slowly varying fast time scale, is generally associated with the WKB Method (after Wentzel, Kramers and Brillouin). Usually, the WKB Assumption (Ansatz, Hypothesis) is restricted to the context of waves, and assumes the solution to be of a particular form. This is further explained below.

For linear wave-type problems we may anticipate the structure of the solution and assume the so-called *WKB Ansatz* or *ray approximation*

$$y(t,\varepsilon) = A(T,\varepsilon)\,\mathrm{e}^{\mathrm{i}\varepsilon^{-1}\int_0^T \omega(\tau,\varepsilon)\,\mathrm{d}\tau}\,. \tag{7.10}$$

The method is again illustrated by the example of the damped oscillator (7.1), but now in complex form, so we consider the real part of (7.10). After substitution and suppressing the exponential factor, we get

$$\begin{aligned}&(1-\omega^2)A + \mathrm{i}\varepsilon\Big(2\omega\frac{\partial A}{\partial T} + \frac{\partial \omega}{\partial T}A + 2\omega A\Big) + \varepsilon^2\Big(\frac{\partial^2 A}{\partial T^2} + 2\frac{\partial A}{\partial T}\Big) = 0,\\ &\mathrm{Re}(A) = 0, \quad \mathrm{Re}(\mathrm{i}\omega A + \varepsilon A') = 0 \ \text{ at } \ T = 0.\end{aligned}$$

Unlike in the multiple scales method the secular terms will not be explicitly suppressed, at least not to leading order. The underlying additional condition here is that the solution of the present type *exists* in the first place and that each higher order correction is no more secular than its predecessor. The solution is expanded as

$$\begin{aligned}A(T,\varepsilon) &= A_0(T) + \varepsilon A_1(T) + \varepsilon^2 A_2(T) + \cdots\\ \omega(T,\varepsilon) &= \omega_0(T) + \varepsilon^2\omega_2(T) + \cdots\,.\end{aligned}$$

Note that ω_1 may be set to zero since the factor $\exp(\mathrm{i}\int_0^T \omega_1(\tau)\,\mathrm{d}\tau)$ may be incorporated in A. By a similar argument, *viz.* by re-expanding the exponential for small ε, all other terms $\omega_2, \omega_3, \ldots$ could be absorbed by A (this is often done). This is perfectly acceptable for the time scale $T = O(1)$, but for larger times we will not be able to suppress higher order secular terms. So we will find it more convenient to include these terms and use them whenever convenient.

We substitute the expansions and collect equal powers of ε to obtain to $O(\varepsilon^0)$

$$(1-\omega_0^2)A_0 = 0$$

with solution $\omega_0 = 1$ (or -1, but that is equivalent for the result). To $O(\varepsilon^1)$ we have then

$$A_0' + A_0 = 0 \ \text{ with } \ \mathrm{Re}(A_0) = 0,\ \mathrm{Im}(\omega_0 A_0) = -1 \text{ at } T = 0,$$

with solution $A_0 = -\mathrm{i}\,\mathrm{e}^{-T}$. To order $O(\varepsilon^2)$ the equation reduces to

$$A_1' + A_1 = -\mathrm{i}(\tfrac{1}{2} + \omega_2)\,\mathrm{e}^{-T}, \ \text{ with } \ \mathrm{Re}(A_1) = 0,\ \mathrm{Im}(\omega_0 A_1) = \mathrm{Re}(A_0') \text{ at } T = 0,$$

with solution

$$\omega_2 = -\tfrac{1}{2}, \quad A_1 = 0.$$

Note that if we had chosen $\omega_2 = 0$, the solution would be $A_1 = -\frac{1}{2}T\,\mathrm{e}^{-T}$. Although by itself correct for $T = O(1)$, it renders the asymptotic hierarchy invalid for $T \geqslant O(1/\varepsilon)$ and is therefore better avoided. The solution that emerges is indeed consistent with the exact solution.

7.1.5 Higher dimensions

In more dimensions, the assumed form of (7.10), where an integral occurs in the argument of the exponential, is not practical. In this case it is more convenient to write

$$\varphi(\boldsymbol{x}, t; \varepsilon) = A(\boldsymbol{X}, T; \varepsilon)\, \mathrm{e}^{\mathrm{i}\varepsilon^{-1}\Omega(\boldsymbol{X},T)}, \tag{7.11}$$

while for clarity of notation we leave Ω independent of ε and introduce the slowly varying frequency and wave vector

$$\omega = \frac{\partial \Omega}{\partial T}, \quad \boldsymbol{\kappa} := -\overline{\nabla}\Omega,$$

where $\overline{\nabla} := \frac{\partial}{\partial X}\boldsymbol{e}_x + \frac{\partial}{\partial Y}\boldsymbol{e}_y + \frac{\partial}{\partial Z}\boldsymbol{e}_z$. This relation implies the continuity equation of waves

$$\frac{\partial \boldsymbol{\kappa}}{\partial T} + \overline{\nabla}\omega = \mathbf{0}. \tag{7.12}$$

Consider the following example of a one-dimensional wave equation with slowly varying coefficients.

$$\frac{\partial}{\partial t}\Big(m(X,T)\frac{\partial}{\partial t}\varphi\Big) = \frac{\partial}{\partial x}\Big(C(X,T)\frac{\partial}{\partial x}\varphi\Big) + B(X,T)\varphi, \tag{7.13}$$

where $X = \varepsilon x$ and $T = \varepsilon t$ are slow variables. We assume the solution φ to take the form given by (7.11). This yields the equation

$$-\omega^2 m A + \frac{\mathrm{i}\varepsilon}{A}\frac{\partial}{\partial T}\big(\omega m A^2\big) = -\kappa^2 C A - \frac{\mathrm{i}\varepsilon}{A}\frac{\partial}{\partial X}\big(\kappa C A^2\big) + BA + O(\varepsilon^2). \tag{7.14}$$

As before, we expand

$$A = A_0 + \varepsilon A_1 + O(\varepsilon^2).$$

After substitution and collecting equal powers of ε, we get from leading order the slowly varying dispersion relation for ω and κ, or eikonal-type equation for Ω

$$\omega^2 m = \kappa^2 C - B. \tag{7.15}$$

Equation (7.12) turns into

$$\frac{\partial \kappa}{\partial T} + V(\kappa)\frac{\partial \kappa}{\partial X} = 0,$$

showing that both κ and ω propagate with the group velocity (section 8.5.1)

$$V = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa} = \frac{\kappa C}{\omega m}. \tag{7.16}$$

The next order yields a conservation-type equation for A_0

$$\frac{\partial}{\partial T}\big(\omega m A_0^2\big) + \frac{\partial}{\partial X}\big(\kappa C A_0^2\big) = 0. \tag{7.17}$$

(It should be noted that this result reflects the underlying physics, and therefore depends on the original equation. In general the resulting equation is not of conserved type.) The pair $\omega m A_0^2$ and $\kappa C A_0^2$ are called *adiabatic invariants*, because they correspond to density and flux of a quantity that is conserved on the slow time and length scales. This is seen as follows. When we integrate (7.17) between the moving boundaries $X = X_1(T)$ and $X = X_2(T)$, we obtain

$$\int_{X_1}^{X_2} \frac{\partial}{\partial T}\big(\omega m A_0^2\big) + \frac{\partial}{\partial X}\big(\kappa C A_0^2\big)\,\mathrm{d}X = \frac{\mathrm{d}}{\mathrm{d}T}\int_{X_1}^{X_2} \omega m A_0^2\,\mathrm{d}X \\ - V_2[\omega m A_0^2]_{X_2} + V_1[\omega m A_0^2]_{X_1} + [\kappa C A_0^2]_{X_2} - [\kappa C A_0^2]_{X_1} = 0$$

where $V_1 = \frac{\mathrm{d}}{\mathrm{d}T}X_1$ and $V_2 = \frac{\mathrm{d}}{\mathrm{d}T}X_2$. This reduces to

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_{X_1}^{X_2} \omega m A_0^2\,\mathrm{d}X = 0,$$

if the velocity of either end point is equal to the group velocity (7.16). In other words, the integral of $\omega m A_0^2$ is conserved between two points moving with the local group velocity.

Suppose, for definiteness, that ϕ denotes position and m mass. Then, if ω and κ are real, we can derive from (7.17) with (7.15) the conservation laws for *wave action* ($\frac{1}{2}\omega m|A_0|^2$ is the wave action density [35])

$$\frac{\partial}{\partial T}\big(\tfrac{1}{2}\omega m|A_0|^2\big) + \frac{\partial}{\partial X}\big(\tfrac{1}{2}\kappa C|A_0|^2\big) = 0 \tag{7.18a}$$

and *energy* ($\frac{1}{2}\omega^2 m|A_0|^2$ is the wave energy density)

$$\frac{\partial}{\partial T}\big(\tfrac{1}{2}\omega^2 m|A_0|^2\big) + \frac{\partial}{\partial X}\big(\tfrac{1}{2}\omega\kappa C|A_0|^2\big) = 0 \tag{7.18b}$$

by substituting $A_0 = |A_0|\exp(\mathrm{i}\arg A_0)$, dividing out the complex exponent, and taking the real part of what remains.

7.1.6 Weakly nonlinear resonance problems

Similar arguments may be applied to certain weakly nonlinear resonance problems. Consider first the following slightly damped[2] harmonic oscillator with harmonic forcing

$$y'' + \omega_0^2 y + 2\omega_0 \sin\theta\, y' = \varepsilon\cos(\omega t), \quad y(0) = y'(0) = 0, \tag{7.19}$$

(with $\sin\theta > 0$ small) which has the solution

$$y(t) = \varepsilon\,\mathrm{e}^{-\omega_0 t\sin\theta}\,\frac{(\omega^2-\omega_0^2)\cos(\omega_0 t\cos\theta) - \tan\theta(\omega^2+\omega_0^2)\sin(\omega_0 t\cos\theta)}{(\omega^2-\omega_0^2)^2 + (2\omega\omega_0\sin\theta)^2} \\ - \varepsilon\frac{(\omega^2-\omega_0^2)\cos(\omega t) - 2\omega\omega_0\sin\theta\sin(\omega t)}{(\omega^2-\omega_0^2)^2 + (2\omega\omega_0\sin\theta)^2}.$$

When $\sin\theta$ is very small, we can distinguish an initial regime where the solution becomes approximately

$$y(t) \simeq \varepsilon\frac{\cos(\omega_0 t) - \cos(\omega t)}{\omega^2-\omega_0^2} = 2\varepsilon\frac{\sin\left(\frac{\omega-\omega_0}{2}t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right)}{\omega^2-\omega_0^2}$$

[2] By writing the friction coefficient $2\omega_0\sin\theta$ in this way, the solution has a neater form.

and the steady state regime for $t \to \infty$ where the solution becomes

$$y(t) \simeq -\varepsilon \frac{\cos(\omega t)}{\omega^2 - \omega_0^2}.$$

We are interested in the behaviour near resonance, when $\omega \approx \omega_0$. Let us assume, for definiteness, that $\omega, \omega_0 = O(1)$ and $\omega - \omega_0 = O(\varepsilon)$. Note that this implies that the factor $\omega^2 - \omega_0^2 = O(\varepsilon)$.

Initially, we have two time scales, *viz.* a fast time $(\omega+\omega_0)t = O(t)$ and a slow time $(\omega-\omega_0)t = O(\varepsilon t)$. As long as $(\omega - \omega_0)t = O(\varepsilon)$, solution y is of the order of magnitude of its driving force, namely $y = O(\varepsilon)$. However, once we are in the steady state, the solution grows an order of magnitude higher and becomes $y = O(1)$.

It is important to realise that near resonance we are not able to assess the order of magnitude of y from the driving force alone. We have to be more careful.

Consider these arguments to obtain (as a typical example) the steady state, near resonance solution of the weakly non-linear, harmonically driven oscillator

$$y'' + \omega_0^2 y + \varepsilon a y^3 = \varepsilon C \cos(\omega t), \quad \omega = \omega_0(1+\varepsilon\sigma), \quad \omega, \omega_0, a, C, \sigma = O(1) \tag{7.20}$$

asymptotically for $\varepsilon \to 0$. (Note that σ and a do not need to be positive.)

In steady state, the solution will follow the periodicity of the driving force and will therefore be periodic with frequency ω. In other words, y will be a function $f(\omega t)$ of ε-dependent argument $\omega(\varepsilon)t$. Like in the Lindstedt-Poincaré and Multiple Scales methods, an asymptotic expansion in powers of ε (assuming a smooth f) will include secular terms like

$$f(\omega t) = f(\omega_0 t) + \varepsilon\sigma t f'(\omega_0 t) + \ldots$$

and so spoil any regular expansion on a time scale larger than $O(1)$. It is therefore better to absorb the ε-dependent ω into a new time variable $\tau = \omega t$. Next to this, it is convenient to rescale a and y

$$\tau = \omega t, \quad a = \frac{\alpha\omega_0^6}{C^2}, \quad y(t) = \frac{C}{\omega_0^2}\phi(\tau)$$

to obtain

$$(1+\varepsilon\sigma)^2 \phi_{\tau\tau} + \phi + \varepsilon\alpha\phi^3 = \varepsilon\cos\tau. \tag{7.21}$$

Neglecting for the moment the non-linear term, we have seen above that away from resonance, y follows the driving force and remains $y = O(\varepsilon)$, but near resonance it grows to become at steady state $y = O(\varepsilon/(\omega - \omega_0) = O(1)$. So we assume $\phi = O(1)$ and assume the Poincaré expansion

$$\phi = \phi_0 + \varepsilon\phi_1 + \ldots$$

which we substitute in the equation. By collecting corresponding orders we obtain in the usual way

$$\phi_0'' + \phi_0 = 0,$$
$$\phi_1'' + \phi_1 + 2\sigma\phi_0'' + \alpha\phi_0^3 = \cos\tau.$$

Initially, ϕ_0 is totally undetermined, and we can say little more than the general solution

$$\phi_0(\tau) = A_0\cos\tau + B_0\sin\tau.$$

We may see that A_0 and B_0 is determined at the next order, but it is not immediately clear how. First order ϕ_1 is driven by both the external force $\cos\tau$ *and* terms inherited from leading order ϕ_0, and we need to know ϕ_0 before we can proceed at all. So the situation looks rather hopeless.

There is, however, information that we haven't used yet. While, on the one hand, ϕ_1 is excited at resonance (by the external force and the $\cos\tau$ and $\sin\tau$ terms from ϕ_0) leading to algebraic growth by secular terms, we are, on the other hand, looking for a steady state solution such that $\varepsilon\phi_1$ remains $O(\varepsilon)$ and does not grow to $O(1)$.

In other words, the secular terms of ϕ_1 should not be present and have to be suppressed. This provides us with the *consistency condition* that yields the missing equations to determine A_0 and B_0.

From the driving terms of ϕ_1

$$\begin{aligned}\phi_1'' + \phi_1 &= \cos\tau - 2\sigma\phi_0'' - \alpha\phi_0^3 \\ &= \big[1 + 2\sigma A_0 - \tfrac{3}{4}\alpha A_0(A_0^2 + B_0^2)\big]\cos\tau + \big[2\sigma B_0 - \tfrac{3}{4}\alpha B_0(A_0^2 + B_0^2)\big]\sin\tau \\ &\qquad - \tfrac{1}{4}\alpha\big[A_0(A_0^2 - 3B_0^2)\cos(3\tau) + B_0(3A_0^2 - B_0^2)\sin(3\tau)\big]\end{aligned}$$

we obtain the conditions that suppress the secular terms

$$\begin{aligned}1 + 2\sigma A_0 - \tfrac{3}{4}\alpha A_0(A_0^2 + B_0^2) &= 0, \\ 2\sigma B_0 - \tfrac{3}{4}\alpha B_0(A_0^2 + B_0^2) &= 0.\end{aligned}$$

It is immediately clear that $B_0 = 0$, while A_0 is found from a (real) root of the 3$^{\text{rd}}$-order polynomial

$$4x^3 - \lambda(3x + 1) = 0, \quad x = \tfrac{2}{3}\sigma A_0, \quad \lambda = \frac{128\sigma^3}{81\alpha} \tag{7.22}$$

We find 1 root for $\lambda < 1$, 2 roots for $\lambda = 1$ and 3 roots for $\lambda > 1$.

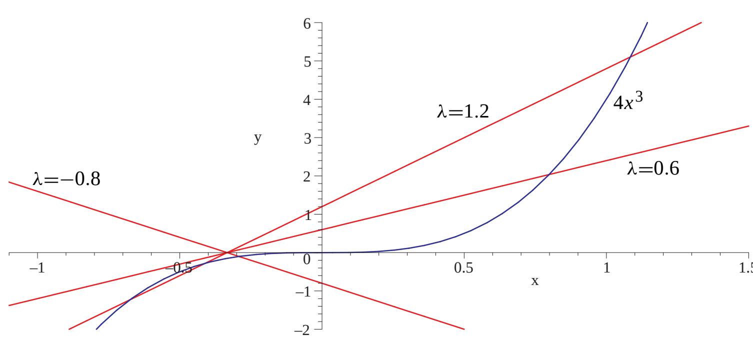


Figure 7.2: Examples of intersection points of $y = 4x^3$ and $y = \lambda(3x + 1)$.

We can go on to determine ϕ_1. Again this will contain undetermined coefficients, which have to be determined at the next order.

7.2 Multiple Scales, WKB and Resonance: Assignments

7.2.1 Non-stationary Van der Pol oscillator

Consider the weakly nonlinear oscillator, described by the Van der Pol equation, for variable $y = y(t, \varepsilon)$ in t:

$$y'' + y - \varepsilon(1 - y^2)y' = 0$$

asymptotically for small positive parameter ε. (Check the phaseplane figure 9.2 in section 9.1.)

Construct by means of the method of multiple scales a first-order approximate solution. You are free to choose convenient (non-trivial) initial values.

7.2.2 The air-damped, unforced pendulum

For sufficiently high Reynolds numbers, the air-damped pendulum may be described by

$$m\ddot{\phi} + C\dot{\phi}\,|\dot{\phi}| + K\sin\phi = 0, \quad \phi(0) = \varepsilon, \;\; \dot{\phi}(0) = 0,$$

where $\varepsilon > 0$ is small and problem parameters m, K and C are positive. Assume that $C/m = O(\varepsilon)$. Use the method of multiple scales to get an asymptotic approximation of $\phi = \phi(t, \varepsilon)$ for $\varepsilon \to 0$.

7.2.3 The air-damped pendulum, harmonically forced near resonance

When an oscillator of resonance frequency ω_0 is excited harmonically, with a frequency ω near ω_0, the resulting steady state amplitude may be much larger than the forcing amplitude. Nonlinear effects may be called into action and limit the amplitude, which otherwise (in the linear model) would have been unbounded at resonance. In the following we will study an air-damped oscillator with harmonic forcing near resonance. The chosen parameter values are such that the resulting amplitude is just large enough to be bounded by the nonlinear damping.

a) Consider the damped harmonic oscillator with harmonic forcing

$$m\ddot{\phi} + K\phi = F\cos(\omega t).$$

Parameters m, K and F are positive. Find the steady state solution, *i.e.* the solution harmonically varying with frequency ω.

b) Consider the air-damped version

$$m\ddot{\phi} + C\dot{\phi}\,|\dot{\phi}| + K\phi = F\cos(\omega t),$$

where problem parameters m, K, C and F are positive. C and F are small in a way that $F = \varepsilon K$ and $C = \varepsilon m\beta$ where ε is small. The resonance frequency of the undamped linearised problem is $\omega_0 = \sqrt{K/m}$, while $\omega/\omega_0 = \Omega = 1 + \varepsilon\sigma$ with detuning parameter $\sigma = O(1)$. We are interested in the (bounded) steady state, and initial conditions are unimportant. Use similar techniques as used with the methods of multiple scales and Lindtedt-Poincaré to get an asymptotic approximation of $\phi = \phi(t)$ for $\varepsilon \to 0$.
Hint: make t dimensionless by $\tau = \omega t$. Write the leading order solution in the form $\phi_0 = A_0\cos(\tau - \tau_0)$ and find A_0 as a function of σ from ϕ_1. Note that $A_0(\sigma) \to 0$ if $\sigma \to \pm\infty$, which is in agreement with (a).

c) The same problem as above but with a nonlinear restoring force $K\sin\phi$, *i.e.*

$$m\ddot{\phi} + C\dot{\phi}\,|\,\dot{\phi}\,| + K\sin\phi = F\cos(\omega t),$$

while we now choose $F = \varepsilon^3 K$, $C = \varepsilon m\beta$ and $\omega/\omega_0 = \Omega = 1 + \varepsilon^2\sigma$. Note that we have to rescale ϕ.
The main difference with (b) is that A_0 cannot be expressed explicitly in σ, but if we plot σ as a function of A_0, we can recognise the physical solutions that satisfy $A_0(\sigma) \to 0$ if $\sigma \to \pm\infty$.

7.2.4 Relativistic correction for Mercury

The relativistic correction in the calculation of the advance of the perihelion of Mercury.
In the relativistic mechanics of planetary motion around the Sun, one comes across the problem of solving

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \alpha(1 + \varepsilon u^2) \quad \text{for} \quad 0 < \theta < \infty,$$

where $u(0) = 1$ and $u'(0) = 0$. Here θ is the angular coordinate in the orbital plane, $u(\theta) = 1/r$, where r is the normalized radial distance of the planet from the Sun, and α is a positive constant. Note that if $\varepsilon = 0$ then one obtains the Newtonian description.

a) Find a first-term approximation of the solution that is valid for large θ.

b) Using the results of part (a), find a two-term expansion of the angle $\Delta\theta$ between successive perihelions, that is, the angel between successive maxima in $u(\theta)$.

c) The parameters in the equation are

$$\varepsilon = 3\Big(\frac{h}{cr_c}\Big)^2, \quad \alpha = \frac{r_c}{a(1 - e^2)},$$

where h is the angular momentum of the planet per unit mass, r_c is a characteristic orbital distance, c is the speed of light, a is the semi-major axis of the elliptic orbit, and e is the eccentricity of the orbit.

For the planet Mercury, $h/c = 9.05 \cdot 10^3$ km, $r_c = a = 57.91 \cdot 10^6$ km, and $e = 0.20563$ (Nobili and Will, 1986). It has been observed that the precession of Mercury's perihelion, defined as $\Delta\phi = \Delta\theta - 2\pi$, after a terrestrial century is $43.11'' \pm 0.45''$ (note that Mercury orbits the sun in 0.24085 years). How does this compare with your theoretical result in (b)?

The problem is classic, and formed one of the famous experimental evidences of Einstein's theory of relativity. Make sure to do the deceivingly trivial calculations correctly. The results will agree!

7.2.5 Weakly nonlinear advection-diffusion

Consider the following advection problem with weak diffusion:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \quad \text{for} \quad -\infty < x < \infty, \quad 0 < t,$$

where $u(x,0) = f(x)$. Using multiple scales, find a first-term approximation of the solution that is valid for large t. Assume (and use) the fact that f is Fourier transformable:

$$\hat{f}(\alpha) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha.$$

Apply the formal result to

$$f(x) = \mathrm{e}^{-\frac{1}{2}x^2}, \qquad \hat{f}(\alpha) = \sqrt{2\pi}\,\mathrm{e}^{-\frac{1}{2}\alpha^2}.$$

7.2.6 Golden Ten: an application of multiple scales

Golden Ten [36] is a modified version of Roulette, played with a small ball moving in a relatively large conical bowl. At the end of the game the ball falls in one of 26 numbered compartments placed along the number ring. In contrast to Roulette, Golden Ten is a so-called observation game: the players have to stake only after the ball has reached a certain level at the bowl. It is claimed that the possibility to observe a part of the orbit of the ball enables the player to make a better than random guess on the outcome.

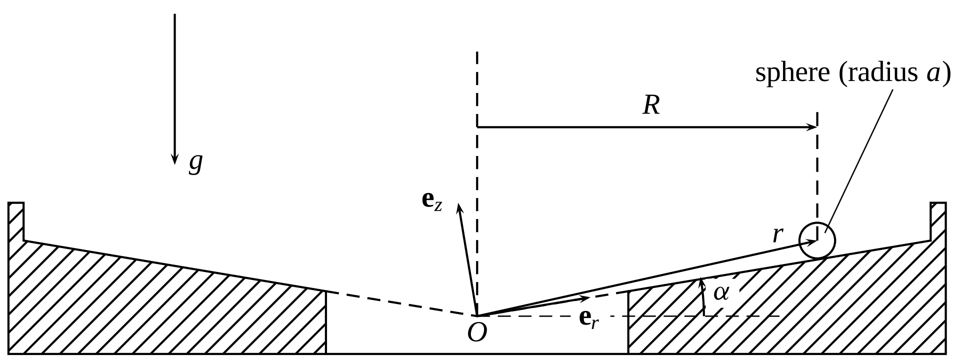


Figure 7.3: The Golden Ten bowl; cross-sectional view

To construct a mechanical model (*equations of motion*) for the motion of the ball the following basic assumptions are made:

i) the ball is a homogeneous sphere (mass m, radius a);

ii) the bowl is rotationally symmetric and purely horizontal (angle of inclination: α);

iii) the ball rolls without slipping (the surface of the bowl is rather smooth, but rough enough to prevent the ball from slipping);

iv) the motion of the ball is completely deterministic (of course, in reality there are, inevitably, random effects, but they are not taken into account in our mechanical model);

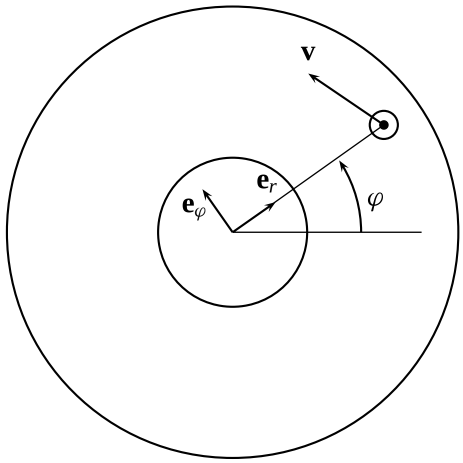


Figure 7.4: Top-view of the bowl with moving sphere

v) the ball is launched along a vertical rim (radius $R_{\rm rim}$) at the top of the bowl (after rolling a few laps alongside the rim the ball smoothly leaves the rim, and gradually spirals down the bowl).

A frame $\{O\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\}$, moving with the ball, is introduced as shown in Fig. 7.4. The origin O is fixed in the apex of the bowl, but the frame rotates with angular velocity $\dot{\varphi}$ about a vertical axis through O, such that the point of contact P between ball and bowl is always on the $\boldsymbol{e}_1$-axis (distance $OP = r$). Hence, the angular velocity of the frame is

$$\boldsymbol{\Omega} = \dot{\varphi} \sin\alpha \boldsymbol{e}_1 + \dot{\varphi} \cos\alpha \boldsymbol{e}_3 \ , \tag{7.23}$$

(note that $\dot{\boldsymbol{e}}_i = \boldsymbol{\Omega} \times \boldsymbol{e}_i$) and the position vector of the centre o of the ball is

$$\boldsymbol{x}_o = r\boldsymbol{e}_1 + a\boldsymbol{e}_3 \ . \tag{7.24}$$

For later use, we introduce the distance R from o to the vertical through O, i.e.

$$R = r\cos\alpha - a\sin\alpha \ . \tag{7.25}$$

The velocity $\boldsymbol{v}_o = \dot{\boldsymbol{x}}_o$ and the acceleration $\boldsymbol{a}_o = \dot{\boldsymbol{v}}_o = \ddot{\boldsymbol{x}}_o$ of o can now be expressed in the variables R and φ and their derivatives (*all further details of the derivations are omitted here*).

The angular velocity $\boldsymbol{\omega}$ of the ball must be derived from the condition that the ball rolls, implying that the velocity of the point P of the ball that is momentarily in contact with the bowl must be zero. This yields

$$\boldsymbol{\omega} = -\frac{R\,\dot{\varphi}}{a}\boldsymbol{e}_1 + \frac{\dot{R}}{a\cos\alpha}\boldsymbol{e}_2 + \dot{\psi}\,\boldsymbol{e}_3 \ , \tag{7.26}$$

where $\dot{\psi} = (\boldsymbol{\omega}, \boldsymbol{e}_3)$ is the so-called *spin*. So the ball rolls in $\boldsymbol{e}_1$-(radially, downwards) and $\boldsymbol{e}_2$-(tangential) direction and spins about the normal on the drum surface.

The *equations of motion* for the ball follow from the law of momentum and the law of moment of momentum (Newton-Euler equations), reading

$$\dot{\boldsymbol{p}} = m\,\ddot{\boldsymbol{x}}_o = \boldsymbol{F}, \quad \text{and} \quad \dot{\boldsymbol{D}} = I\,\dot{\boldsymbol{\omega}} = \boldsymbol{M}_o\ , \tag{7.27}$$

where $I = \frac{2}{5}ma^2$, the central moment of inertia of the ball. Furthermore, $\boldsymbol{F}$ and $\boldsymbol{M}_o$ are the total force and the total moment about o on the ball, respectively.

Four distinct forces act on the ball:

i) the gravitational force in o

$$\boldsymbol{F}_g = -mg\sin\alpha\,\boldsymbol{e}_1 - mg\cos\alpha\,\boldsymbol{e}_3\ ; \tag{7.28}$$

ii) the resistive force in o due to air friction (a *linear* air resistance model is chosen here, so the coefficient f is constant)

$$\boldsymbol{F}_a = -mf\,\boldsymbol{v}_o\ , \tag{7.29}$$

(the coefficient of resistivity is written as: mf for convenience);

iii) the normal force in P

$$\boldsymbol{F}_n = N\boldsymbol{e}_3, \quad (N > 0)\ ; \tag{7.30}$$

iv) the frictional force in P due to dry friction

$$\boldsymbol{F}_d = D_1\boldsymbol{e}_1 + D_2\boldsymbol{e}_2\ ; \tag{7.31}$$

(here: N, D_1 and D_2 are unknown).

Note that of these four forces only $\boldsymbol{F}_d$ contributes to $\boldsymbol{M}_o$ by a moment equal to

$$\boldsymbol{M}_d = (-a\boldsymbol{e}_3 \times \boldsymbol{F}_d) = aD_2\boldsymbol{e}_1 - aD_1\boldsymbol{e}_2\ .$$

We neglect resistive moments due to rolling and spinning of the ball. Thus,

$$\boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{F}_a + \boldsymbol{F}_n + \boldsymbol{F}_d, \quad \boldsymbol{M}_o = \boldsymbol{M}_d\ .$$

With use of the preceding results in the Newton-Euler equations (7.27) and after the elimination of the unknowns N, D_1 and D_2, the following three equations of motion for $R(t)$, $\varphi(t)$ and $\psi(t)$ emerge

$$\ddot{R} = -\tfrac{5}{7}f\,\dot{R} + R\,\dot{\varphi}^2\cos^2\alpha + \tfrac{2}{7}a\,\dot{\varphi}\dot{\psi}\,\sin\alpha\cos\alpha - \tfrac{5}{7}g\cos\alpha\sin\alpha\ , \tag{7.32a}$$

$$\ddot{\varphi} = -\tfrac{5}{7}f\,\dot{\varphi} - \frac{2}{R}\,\dot{R}\dot{\varphi}\ , \tag{7.32b}$$

$$\ddot{\psi} = -\frac{1}{a}\,\dot{R}\dot{\varphi}\,\tan\alpha\ . \tag{7.32c}$$

For the *initial conditions* we assume that the ball rolls along the rim for $t < 0$, and looses contact with the rim at $t = 0$ (smoothly). When the ball rolls along the rim, as well as on the bowl, the following relation must hold

$$a\,\dot{\psi}\,\cos\alpha + R\,\dot{\varphi}\,(1 - \sin\alpha) = 0\ , \quad (\text{for } t \leqslant 0)\ . \tag{7.33}$$

At the moment of loosening $t = 0$ there is no force acting between the ball and the rim. Hence, at $t = 0$ it is as if the ball moves, *momentarily*, in a circular orbit with

$$R = R_0 = R_{\text{rim}} - a, \quad \dot{R} = \ddot{R} = 0, \quad \dot{\varphi} = \omega_o, \quad \dot{\psi} = \Omega_0 ,$$

with ω_0 and Ω_0 still unknown.

From these considerations the following set of initial conditions can be derived (*the details are left to the student*)

$$\begin{aligned}
&R(0) = R_0 = R_{\text{rim}} - a, \quad \dot{R}\ (0) = 0 , \\
&\varphi(0) = 0, \quad \dot{\varphi}\ (0) = \omega_0 = \sqrt{\frac{5g\sin\alpha}{R_0(7\cos\alpha - 2(1-\sin\alpha)\tan\alpha)}} , \\
&\dot{\psi}\ (0) = \Omega_0 = -\frac{R_0\omega_0}{a\cos\alpha}(1-\cos\alpha) .
\end{aligned} \tag{7.34}$$

Here, $\varphi(0)$ is arbitrarily chosen zero, because only the relative difference $(\varphi(t) - \varphi(0))$ is relevant.

With (7.32) and (7.34) the motion of the ball is completely described. These equations can not be solved analytically; only by numerical integration $R(t)$, $\varphi(t)$ and $\psi(t)$ can be determined. Here, we shall try to find some asymptotic results: one for the total path of the ball from the rim to the number ring and a more local one, restricted to one orbit ("ellipse") of the ball around the vertical axis.

parameter	value	unit	parameter	value	unit
m	0.0383	kg	a	0.0175	m
R_{rim}	0.487	m	R_0	0.469	m
R_{num}	0.205	m	α	0.0831	rad
g	9.81	m/sec^2	f	0.014	sec^{-1}
total time for one game $t_f \approx 116$ sec					

Table 7.1: Numerical values for parameters of Golden Ten.

Normalization of the equations of motion

If $f = 0$ (no air resistance) the equations of motion (7.32) permit the following two first integrals:

$$R^2\,\dot{\varphi} = C_1 = R_0^2\omega_0 , \tag{7.35}$$

and

$$a\,\dot{\psi}\, - R\,\dot{\varphi}\,\tan\alpha = C_2 = \frac{R_0\omega_0}{\cos\alpha} , \tag{7.36}$$

(both are examples of conservation of moment of momentum, the first being Kepler's law; also the total mechanical energy is conserved, but we shall not use this here).

We should note that if f is positive but small, the changes in the functions introduced in the left-hand sides of (7.35) and (7.36) will be small too. Therefore, we introduce the new variables

$$y_1(t) = \frac{R^2\,\dot{\varphi}}{R_0^2\omega_0}, \quad y_2(t) = \frac{-a\,\dot{\psi}\,\cos\alpha + R\,\dot{\varphi}\,\sin\alpha}{R_0\omega_0}, \quad y_3(t) = \frac{R(t)}{R_0} . \tag{7.37}$$

In observations of the real motion of the ball (i.e. in playing Golden Ten) the angle φ is the more natural variable compared to the time t. Therefore, let us replace the variable t by φ, by use of

$$\frac{d}{dt} = \dot{\varphi}\,\frac{d}{d\varphi} = \omega_0 \frac{u^2}{v}\frac{d}{d\varphi}\ , \tag{7.38}$$

where

$$u(\varphi) = \frac{1}{y_3}, \quad \text{and} \quad v(\varphi) = \frac{1}{y_1}\ . \tag{7.39}$$

Finally, we call $y_2 = w(\varphi)$, and we introduce the new dimensionless parameters (which both are small according to Table 7.1)

$$\varepsilon = \frac{5f}{7\omega_0}, \quad \delta = \sin\alpha\ . \tag{7.40}$$

With all this (and with $g\sin\alpha\cos\alpha = \omega^2 R_0(7 - 2\sin\alpha - 5\sin^2\alpha)/5$) the system (7.32)-(7.34) can be rewritten as

$$\begin{aligned} &\frac{dv}{d\varphi} = \varepsilon\frac{v^2}{u^2}\ ,\\ &\frac{dw}{d\varphi} = -\delta\varepsilon\frac{1}{u}\ ,\\ &\frac{d^2u}{d\varphi^2} = -(1-\tfrac{5}{7}\delta^2)u + (1-\tfrac{2}{7}\delta-\tfrac{5}{7}\delta^2)\frac{v^2}{u^2} + \tfrac{2}{7}\delta v w\ ,\\ &u(0) = v(0) = w(0) = 1, \quad \frac{du}{d\varphi}(0) = 0\ . \end{aligned} \tag{7.41}$$

In this form the system is adequate for asymptotics. However, in order to keep the now following two exercises manageable (and *only* for that reason!) we shall neglect in (7.41) all terms containing δ. This results in the following reduced system, in which also the influence of the spin, represented by $w(\varphi)$, is disappeared,

$$\begin{aligned} &\frac{dv}{d\varphi} = \varepsilon\frac{v^2}{u^2}, \quad v(0) = 1\ ,\\ &\frac{d^2u}{d\varphi^2} + u - \frac{v^2}{u^2} = 0, \quad u(0) = 1, \quad \frac{du}{d\varphi}(0) = 0\ . \end{aligned} \tag{7.42}$$

Although this system is a (too) strong simplification of (7.41), we will see that it still contains most of the characteristic features of the motion of the ball in the bowl.

Solve the above system of equations by a multiple scales analysis (to leading orders only).

7.2.7 Modal sound propagation in slowly varying ducts

Consider the problem of sound propagation in a cylindrical duct of slowly varying cross section and slowly varying sound speed. The radius is given by $r = R(\varepsilon x)$, and the sound speed is given by $c_0 = c_0(\varepsilon x)$.

Note that sound speed c_0, mean pressure p_0 and mean density ρ_0 are related by $\rho_0 c_0^2 = 1.4 \cdot p_0$ (in air). The mean pressure is under usual atmospheric circumstances constant. Therefore, the mean density is also slowly varying.

The walls of the duct are soft and sound absorbing, as the wall is an impedance wall, also with (in axial direction) slowly varying impedance. We consider sound waves of a fixed frequency ω and rewrite the sound pressure by introducing the complex pressure p as

$$\text{physical sound pressure} = \mathrm{Re}(p(x, r, \vartheta)\,\mathrm{e}^{\mathrm{i}\omega t}).$$

The modified reduced wave equation for p is

$$\nabla \cdot \left(c_0^2 \nabla p\right) + \omega^2 p = 0.$$

The impedance boundary condition is (rewritten to the pressure)

$$\left(\nabla p \cdot \vec{\boldsymbol{n}}\right) = -\frac{\mathrm{i}\omega\rho_0}{Z} p \qquad \text{at} \quad r = R$$

with $Z = Z(\varepsilon x)$ the complex impedance of the wall, and the normal vector is given by

$$\vec{\boldsymbol{n}} = \frac{\vec{\boldsymbol{e}}_r - \varepsilon R' \vec{\boldsymbol{e}}_x}{\sqrt{1 + \varepsilon^2 R'^2}}$$

It is convenient to introduce the slowly varying function $\zeta = -\mathrm{i}\omega\rho_0 / Z$.

i) Observe that for a straight duct with uniform mean flow and walls ($\varepsilon = 0$) the sound field can be written as a sum over modes, given by

$$\psi_{m\mu} = J_m(\alpha_{m\mu} r)\,\mathrm{e}^{-\mathrm{i}m\vartheta}\,\mathrm{e}^{-\mathrm{i}k_{m\mu} x}$$

where m and $\alpha_{m\mu}$ are so-called eigenvalues (they are eigenvalues of the transverse Laplace problem, but this is not important here). m is an integer, while $\alpha_{m\mu}$ satisfies $\alpha J_m'(\alpha R) = \zeta J_m(\alpha R)$. The axial wave number $k_{m\mu}$ can be expressed in ω / c_0 and $\alpha_{m\mu}$.

ii) Consider now the multiple scales problem for small ε of a slowly varying mode propagating through the duct. Determine analogous to the example in the SIAM book the shape of such a quasi-ductmode to leading order.

Eventually, the following integral of Bessel functions can be used:

$$\int J_m(\alpha r)^2 r \,\mathrm{d}r = \tfrac{1}{2}\left(r^2 - \frac{m^2}{\alpha^2}\right) J_m(\alpha r)^2 + \tfrac{1}{2} r^2 J_m'(\alpha r)^2.$$

7.2.8 A nearly resonant weakly nonlinear forced harmonic oscillator

Consider the system governed by the equation of motion

$$y'' + y + \alpha y^3 = \varepsilon^{3/2} \cos \omega t.$$

We are interested in the stationary solution due to the driving force, so initial conditions are not important and solutions of the homogeneous equation are ignored. Find, asymptotically for $\varepsilon \to 0$ and $\alpha = O(1)$, the solution up to second order.

a) for $\omega^2 = 1 + O(1)$, *i.e.* away from resonance;

b) for $\omega^2 = 1 + \varepsilon\mu$, *i.e.* near resonance.

Hint: For part (b), introduce the variable $\tau = \omega t$ and use similar techniques as encountered with the method of multiple scales and Lindstedt-Poincaré.

7.2.9 A non-linear beam with small forcing

The equation of a non-linear beam with a small forcing is

$$\frac{\partial^4}{\partial x^4} u - \kappa \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial t^2} u = f(t) \sin(\pi x)$$

for $0 < x < 1$ and $t > 0$, where $u = \frac{\partial^2}{\partial x^2} u = 0$ at $x = 0, x = 1$. The (time dependent) coefficient κ is defined by

$$\kappa = \frac{1}{4} \int_0^1 \left(\frac{\partial}{\partial x} u \right)^2 \mathrm{d}x.$$

Assume $u(x, t, \varepsilon) = U(t, \varepsilon) \sin(\pi x)$.

a) Find the first-term of an asymptotic expansion for small ε of the solution for $f(t) = \varepsilon \sin(t)$. We do not apply any initial conditions but assume that the solution consists only of the part that is driven by the source $f(t)$.

b) Using again $f(t) = \varepsilon \sin(t)$, solve as multiple scales problem the first two terms of an expansion of the solution, satisfying the initial conditions $u(x, 0) = \frac{\partial}{\partial t} u(x, 0) = 0$.
Hint: note that any combination of the type $\sin(\pi^2 t) \sin(t) = \frac{1}{2} \cos(\pi^2 - 1)t - \frac{1}{2} \cos(\pi^2 + 1)t$ will never be in resonance with vibrations of a frequency of any multiple of π^2 because π^2 is an irrational number.

c) If we take for the driving force $f(t) = \varepsilon^{\frac{3}{2}} \sin(\pi^2 + \omega_0 \varepsilon)t$, find the first-term of an asymptotic expansion of the solution that valid for large t. The general solution of the slow variable problem is difficult to find. Consider only stationary solutions. Can you determine the type of stability of these stationary points?

7.2.10 Acoustic rays in a medium with a varying sound speed

Show that acoustic rays follow circular paths if the sound speed varies linearly in space:

(a) Rewrite the eikonal equation (♯) of Example 15.37 in characteristic form by using Theorem (12.6).

(b) Prove that in a medium with a linearly varying sound speed the path of rays are circles.

Hint: make sure that the parameter s, along the curve $\boldsymbol{x} = \boldsymbol{\xi}(s)$ that represents the ray, corresponds with the curve arc length. In that case $\boldsymbol{t} = \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\xi}$ is the unit tangent vector and $\boldsymbol{\kappa} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\boldsymbol{\xi}$ is the curvature vector. Assume that c_0 varies linearly in direction $\boldsymbol{n}$, i.e. $c_0 = q + \alpha(\boldsymbol{x} \cdot \boldsymbol{n})$. Show that $\boldsymbol{t}\times\boldsymbol{\kappa}$ is constant, and conclude that the torsion is zero and the curve lies in one plane. Show that $|\boldsymbol{\kappa}|$ is a constant, and conclude that the curve is a circle.

7.2.11 Homogenisation as a Multiple Scales problem

Consider a slow flow (like groundwater) or diffusion of matter in a medium with a fine local structure, of which the properties (porosity etc.) vary slowly on a larger scale. Usually we are eventually interested in the large scale behaviour. In this case it makes sense to separate the small and large scales, and see if the effect of the small scale behaviour can be represented by a large scale medium property, by way of a local averaging process of the small scale medium properties. This approach is called *homogenisation*, and can be considered as an application of the method of multiple scales.

Take the following model-problem of diffusion of a concentration u in a medium with a fast varying property a, driven by a slowly varying external source f.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}x}u\right) = f(x).$$

a varies quickly (in x/ε) with a slowly (in x) varying averaged value. For definiteness we will assume a to be of a particular form. Introduce the slow variable $Z = x$ and the fast variable $z = x/\varepsilon$, such that $Z = \varepsilon z$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{a(z,\varepsilon)}\frac{\mathrm{d}}{\mathrm{d}z}u(z,\varepsilon)\right) = \varepsilon^2 f(\varepsilon z)$$

A more general theory is possible for $a(z,\varepsilon) = \alpha(Z) + \beta(z, Z)$ such that

$$\int_0^z \beta(\tau, Z)\,\mathrm{d}\tau = \text{integrable in } z \text{ for } z \to \infty.$$

For the moment we start with assuming α is constant and $\beta = \beta(z)$. Assume the existence of the regular (= uniform Poincaré) asymptotic expansion in the <u>independent</u> variables z and Z

$$u(z,\varepsilon) = U(z, Z, \varepsilon) = U_0(z, Z) + \varepsilon U_1(z, Z) + \varepsilon^2 U_2(z, Z) + \ldots$$

The crucial condition (a form of suppression of secular terms) is that regularity implies a uniform asymptotic sequence of the terms, so $U_0, U_1, U_2, \cdots = O(1)$ for $Z \leqslant O(1)$ and $z \leqslant O(1/\varepsilon)$.

Note: usually this is not uniform on an interval with boundary conditions. At the ends we will have boundary layers $x = O(\varepsilon)$. These will be ignored here.

Derive the following homogenised equation in the slow variable only

$$U_0''(Z) = \alpha f(Z).$$

Indicate how to proceed for higher orders.

7.2.12 The non-linear pendulum with slowly varying length

Consider a pendulum, moving in the (x, y)-plane, of a mass m that is connected to a hinge at $(0, 0)$ by an idealised massless rod of length L, which is varied slowly in time (slow compared to the typical frequency of the fixed-length system). Denote by θ the angle between the rod and the vertical.

At time t the position (x, y), velocity (x', y') and acceleration (x'', y'') of the mass are given by

$$\begin{aligned} x &= L\sin\theta, & x' &= L\theta'\cos\theta + L'\sin\theta, & x'' &= L\theta''\cos\theta + 2L'\theta'\cos\theta - L\theta'^2\sin\theta + L''\sin\theta, \\ y &= -L\cos\theta, & y' &= L\theta'\sin\theta - L'\cos\theta, & y'' &= L\theta''\sin\theta + 2L'\theta'\sin\theta + L\theta'^2\cos\theta - L''\cos\theta. \end{aligned}$$

The balancing forces are then inertia, equal to m times the acceleration, gravity gm in downward y-direction, and a reaction force mR in the direction of the rod. If we regroup the forces in tangential and longitudinal direction and divide by m, we obtain the equations

$$\begin{aligned} L\theta'' + 2L'\theta' + g\sin\theta &= 0, \\ L'' - L\theta'^2 - g\cos\theta &= R. \end{aligned}$$

In the following we will try to find $\theta(t)$ as a function of time when $L(t)$ is given, *i.e.* the first equation. Note that reaction force $R(t)$ then follows straightaway and is *not* part of the problem.

a) Assume that L is of the order of some L_0, θ is of the order of $\sqrt{\varepsilon}$, where small parameter ε is equal to the ratio between the inherent time scale of the pendulum $\sqrt{L_0/g}$ and the inherent time scale (say, λ) of the variations of L. In other words:

$$L := L\Big(\frac{t}{\lambda}\Big), \qquad \varepsilon = \frac{\sqrt{L_0/g}}{\lambda}$$

Make the problem dimensionless, scale the variables in an appropriate way, and expand the equations up to and including terms of $O(\theta^3)$.

b) Solve for $\theta = \theta(t)$ asymptotically for small ε by the WKB method.
Note: don't use the WKB-Ansatz given in equation (7.10) on page 113, because the problem is not linear. Apply Multiple Scales with a slowly varying fast time scale.

7.2.13 Asymptotic behaviour of solutions of Bessel's equation

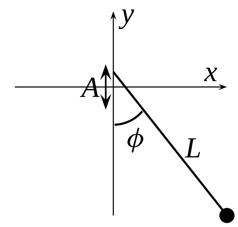
The equation

$$y'' + \frac{1}{r}y' + \Big(\alpha^2 - \frac{m^2}{r^2}\Big)y = 0$$

has solutions in the form of Besselfunctions of order m and argument αr.
Find asymptotic solutions of WKB-type for $\alpha \to \infty$ and $r = O(1)$ with $r > m/\alpha$.
Consider (i) $m^2 = O(1)$, (ii) $m^2 = O(\alpha)$ and (iii) $m^2 = O(\alpha^2)$.

7.2.14 Kapitza's Pendulum

Denote the vertical axis as y and the horizontal axis as x so that the motion of the pendulum happens in the (x, y)-plane. The following notation will be used: ω and A are the driving frequency and amplitude of the vertical oscillations of the suspension, g is the acceleration of gravity, L is the length of the rigid and light pendulum, m is the mass of the bob and $\omega_0 = \sqrt{g/L}$ is the frequency of the free pendulum.



Denoting the angle between pendulum and downward direction as ϕ, the position $x = \xi$, $y = \eta$ of the pendulum at time t is

$$\xi(t) = L \sin\phi$$
$$\eta(t) = -L \cos\phi - A\cos\omega t$$

The potential energy of the pendulum due to gravity is defined by its vertical position as

$$E_{\text{pot}} = -mg(L\cos\phi + A\cos\omega t)$$

The kinetic energy in addition to the standard term $\frac{1}{2}mL^2\,\dot{\phi}^2$ describing the velocity of a mathematical pendulum is the contribution due to the vibrations of the suspension

$$E_{\text{kin}} = \tfrac{1}{2}mL^2\,\dot{\phi}^2 + mAL\omega\sin(\omega t)\sin(\phi)\,\dot{\phi} + \tfrac{1}{2}mA^2\omega^2\sin^2(\omega t)$$

The total energy is then $E = E_{\text{kin}} + E_{\text{pot}}$ and the Lagrangian is $\mathcal{L}(t, \phi, \dot{\phi}) = E_{\text{pot}} - E_{\text{kin}}$. The motion of the pendulum satisfies the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi}$$

which is

$$\ddot{\phi} = -L^{-1}(g + A\omega^2\cos\omega t)\sin\phi.$$

Assume that the driving amplitude A is small compared to L and frequency ω is large compared to the free frequency ω_0, in such a way that $A\omega/L\omega_0 = O(1)$. We make this explicit by writing $\varepsilon = \omega_0/\omega$ and $A/L = \varepsilon\mu$. If we rescale $\tau = \omega t$, we obtain

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -(\varepsilon^2 + \varepsilon\mu\cos\tau)\sin\phi.$$

From the structure of the equation we may infer that $\phi = \phi(\tau, T, \varepsilon)$ has a fast timescale τ and a slow timescale $T = \varepsilon\tau$. Finish the analysis by assuming that ϕ can be written as the sum of a slowly varying large part and a fast varying small part

$$\phi(\tau, T, \varepsilon) = \phi_0(T) + \varepsilon\phi_1(\tau, T) + \varepsilon^2\phi_2(\tau, T) + \ldots$$

Apply a consistency condition for ϕ_2 being bounded for $\tau \to \infty$. Find an equation for ϕ_0 and an expression for ϕ_1. Under what condition on μ are there two stationary solutions ϕ_0? Try to analyse the stability in $\phi_0 = \pi$, the inverted pendulum.

7.2.15 Doppler effect of a moving sound source

The observed pitch of a moving sound source of frequency ω_0 is higher if the source approaches the observer and lower if it recedes from it. This frequency shift, called *the Doppler effect*, occurs if the time scale of the tone ω_0^{-1} is much smaller than the time scale T of the motion, *i.e.* if $\omega_0 T \gg 1$.

a) Consider for smooth amplitude A and phase $\omega_0 T\phi$ the slowly varying, *almost* harmonic signal p

$$p(t) = A(t/T)\,\mathrm{e}^{\mathrm{i}\omega_0 T\phi(t/T)}, \quad \omega_0 T \gg 1.$$

Its *Short Time Fourier Transform* (STFT) P is given by

$$P(\omega;\tau,\sigma) = \int_{-\infty}^{\infty} w(t-\tau,\sigma)\,p(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t,$$

where window function $w(t,\sigma)$ is a non-negative real function symmetric in t around 0, such that it tends to zero fast enough outside of an interval of characteristic width σ. More precisely, we will assume that $w(t,\sigma) \to 1$ for $\sigma \to \infty$ and $w(\sigma\xi,\sigma) \to 0$ for $|\xi| \to \infty$.

Numerically convenient is the rectangular window $w(t,\sigma) = 1$ for $|t| \leqslant \sigma$ and $= 0$ elsewhere. We will use here the analytically more convenient choice, that avoids high-frequent artefacts in P, of Gaussian window $w(t,\sigma) = \mathrm{e}^{-t^2/\sigma^2}$, for which the STFT is called the Gabor transform.

The idea is that for small, but not too small σ we are able to filter out a time dependent Fourier-type spectrum associated to the higher frequencies ($\sim \omega_0$) in signal p. In the present case, with a slow time $O(T)$ of the amplitude and a fast time $O(\omega_0^{-1})$ of the phase, a suitable choice is $\sigma = \sqrt{T/\omega_0}$.

In order to single out in t the relevant σ-neighbourhood of τ we transform $t = \tau + \sigma z$, where $z = O(1)$. Introduce the small parameter $\varepsilon = (\omega_0 T)^{-1}$ and make times t and τ dimensionless on the short time scale. Obtain a form of p reminiscent of the WKB Ansatz for slowly varying almost harmonic functions. Find a small-ε approximation of P, and understand why $\omega_0\phi'$ is indeed sometimes called the *instantaneous frequency*.

b) If the sound field $p(\boldsymbol{x},t)$ of a time-harmonic point source, of frequency ω_0 and moving subsonically along the path $\boldsymbol{x} = \boldsymbol{x}_S(t)$, is given by the equation

$$c_0^{-2}\tfrac{\partial^2}{\partial t^2}p - \nabla^2 p = 4\pi q_0\,\mathrm{e}^{\mathrm{i}\omega_0 t}\,\delta(\boldsymbol{x} - \boldsymbol{x}_S(t)),$$

then the solution in free field is given by the so-called Liénard-Wiechert potential

$$p(\boldsymbol{x},t) = \frac{q_0\,\mathrm{e}^{\mathrm{i}\omega_0 t_e}}{R_e(1 - M_e\cos\theta_e)}$$

where $t_e = t_e(\boldsymbol{x},t)$ is the emission time. This is the time of emission of the signal that travelled along a straight line with the sound speed c_0 from the source in $\boldsymbol{x}_S$ at time t_e to the observer in $\boldsymbol{x}$ at time t. It is a function of $\boldsymbol{x}$ and t, implicitly given by the equation

$$t = t_e + \big\|\boldsymbol{x} - \boldsymbol{x}_S(t_e)\big\|c_0^{-1}.$$

For subsonically moving sources, this equation has exactly one solution. Furthermore, the distance (at emission time) R_e between source and observer, the Mach number M_e of the source speed, and the angle θ_e between the observer direction and the source velocity, are functions of t_e and given by

$$R_e = \left\| \boldsymbol{x} - \boldsymbol{x}_S(t_e) \right\|, \quad M_e = \frac{\left\| \dot{\boldsymbol{x}}_S(t_e) \right\|}{c_0}, \quad \cos\theta_e = \frac{(\boldsymbol{x} - \boldsymbol{x}_S(t_e)) \cdot \dot{\boldsymbol{x}}_S(t_e)}{\left\| \boldsymbol{x} - \boldsymbol{x}_S(t_e) \right\| \left\| \dot{\boldsymbol{x}}_S(t_e) \right\|}.$$

Assuming that time variations due to the ω_0 are much larger than those due to the varying source position, what is the instantaneous frequency observed at position $\boldsymbol{x}$ and time τ?

7.2.16 Vibration modes in a slowly varying elastic beam

Small lateral deflections $u(s,t)$ of a slender beam (a so-called Rayleigh beam) of density ρ, Young's modulus E, slowly varying cross sectional area $A(s)$ and slowly varying moment of inertia $I(s)$, is described by the equation

$$\rho A \frac{\partial^2 u}{\partial t^2} - \rho \frac{\partial}{\partial s}\left(I \frac{\partial^3 u}{\partial t^2 \partial s} \right) + E \frac{\partial^2}{\partial s^2}\left(I \frac{\partial^2 u}{\partial s^2} \right) = 0.$$

Assume for convenience a beam with $A(s) = D(s)^2$ and $I(s) = D(s)^4$.

a) Consider a straight bar, *i.e.* a configuration without slowly varying geometry. Investigate the possible harmonic waves $u(s,t) = U\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}ks}$. What is k, for given ω?

b) Consider the varying bar. Assume a frequency ω such, that the typical corresponding *real* wave length is of the order of magnitude of a diameter. Verify that this corresponds with $kD_0 = O(1)$ and $\omega^2 = O(E/\rho D_0^2)$.

   Derive the differential equation for waves of the form $u(s,t) = U(s)\,\mathrm{e}^{\mathrm{i}\omega t}$ along the beam. Make the problem dimensionless on ρ, E and a typical diameter D_0. Write $s = D_0 z$, $A = D_0^2 a$, $I = D_0^4 a^2$.

c) In axial direction, the beam parameters vary with length scale L which is much longer than D_0. Introduce the slenderness $\varepsilon = D_0/L \ll 1$. We have thus the slowly varying $a = a(\varepsilon z)$. Find a WKB approximation of $U(z) = \Phi(\varepsilon z) \exp\left(\mathrm{i}\varepsilon^{-1} \int^{\varepsilon z} \kappa(\xi)\mathrm{d}\xi\right)$.

7.2.17 An aging spring

A mass $M = M(t)$ is attached to a spring, with spring coefficient $K = K(t)$. The position $u = u(t)$ is given by the equation

$$(Mu')' + Ku = 0$$

a) Assume that $M = M_0$ is constant and the spring is slowly aging according to $K(t) = K_0\,\mathrm{e}^{-\alpha t}$. Make t dimensionless on the inherent time scale T of the oscillator when $t \approx 0$. In order to concretise the (relative !) slowness of the aging, we assume that T is much smaller than $1/\alpha$ and introduce the small parameter $\varepsilon = \alpha T$. Solve the resulting equation asymptotically for $\varepsilon \to 0$ by using the WKB method.

b) The same question for a constant spring coefficient $K = K_0$ and a mass, slowly decaying according to $M(t) = M_0\,\mathrm{e}^{-\alpha t}$.

Chapter 8

Integral Asymptotics

8.1 Integrals and Watson's Lemma

In the following sections we will consider methods to determine the asymptotic behaviour of functions defined by integrals. From Section 3.2 item 18 we know that a uniform asymptotic approximation can be integrated directly. For a non-uniform approximation the situation is more subtle. An example is the following integral. The result is important in its own right but in particular the proof is typical and interesting.

Theorem 8.1 (A Result for Cauchy integrals) *Let f be locally integrable*[1] *in $\mathbb{R}$, such that there is a $p > 0$ with*

$$f(t) = O(|t|^{-p}) \quad \textit{for} \quad t \to \pm\infty.$$

Then the following Cauchy-type integral in $z \in \mathbb{C}$, $z \notin \mathbb{R}$ has the asymptotic behaviour

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,\mathrm{d}t = \begin{cases} O(z^{-p}) & \textit{if} \quad 0 < p < 1, \\ O(z^{-1}\log z) & \textit{if} \quad p = 1, \\ O(z^{-1}) & \textit{if} \quad p > 1, \end{cases} \qquad \textit{for} \quad |z| \to \infty, \quad |\arg(\pm z)| \geqslant \delta > 0.$$

Proof

We consider the right half-range integral on $(0, \infty)$ first. The left half is analogous.
By definition there are numbers K and t_0 such that

$$|f(t)| \leqslant K t^{-p} \quad \text{for} \quad t > t_0.$$

We split up the integral and apply the above.

$$\left|\int_0^{\infty} \frac{f(t)}{t-z}\,\mathrm{d}t\right| \leqslant \int_0^{\infty} \left|\frac{f(t)}{t-z}\right| \mathrm{d}t \leqslant \int_0^{t_0} \frac{|f(t)|}{|t-z|}\,\mathrm{d}t + \int_{t_0}^{\infty} \frac{K}{t^p|t-z|}\mathrm{d}t$$

Write $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ and assume that $r \geqslant 2t_0$ such that along $[0, t_0]$ we have $|t-z| \geqslant r - t \geqslant \frac{1}{2}r$, then

$$\int_0^{t_0} \frac{|f(t)|}{|t-z|}\,\mathrm{d}t \leqslant \frac{2}{r}\int_0^{t_0} |f(t)|\,\mathrm{d}t.$$

[1] $|f|$ is integrable on any finite interval.

If $0 < p < 1$, then t^{-p} is integrable at $t = 0$. We can estimate $|t - z| \geqslant \left|\sin\tfrac{1}{2}\theta\right| (t + r)$. Then

$$\int_{t_0}^{\infty} \frac{K}{t^p|t-z|}\mathrm{d}t \leqslant \int_0^{\infty} \frac{K}{t^p|t-z|}\mathrm{d}t \leqslant \frac{K}{r^p\left|\sin\frac{1}{2}\theta\right|}\int_0^{\infty}\frac{1}{\tau^p(\tau+1)}\mathrm{d}\tau = \frac{K}{r^p\left|\sin\frac{1}{2}\theta\right|}\frac{\pi}{\sin(\pi p)}.$$

If $p > 1$, we find similarly

$$\int_{t_0}^{\infty} \frac{K}{t^p|t-z|}\mathrm{d}t \leqslant \frac{K}{r\left|\sin\frac{1}{2}\theta\right|}\int_{t_0}^{\infty}\frac{1}{t^p}\mathrm{d}t = \frac{K}{r\left|\sin\frac{1}{2}\theta\right|}\frac{t_0^{1-p}}{p-1}.$$

If $p = 1$, we find

$$\int_{t_0}^{\infty} \frac{K}{t|t-z|}\mathrm{d}t \leqslant \frac{K}{r\left|\sin\frac{1}{2}\theta\right|}\int_{t_0}^{\infty}\Big(\frac{1}{t}-\frac{1}{t+r}\Big)\mathrm{d}t = \frac{K}{r\left|\sin\frac{1}{2}\theta\right|}\Big(\log r + \log\big(t_0^{-1}+r^{-1}\big)\Big).$$

The results follow now immediately. □

Another important type is the following Laplace integral[2]. If $f(t)$ is N times continuously differentiable on $[0, \infty)$ and bounded, then the main contribution for $\varepsilon \to 0$ of

$$\int_0^{\infty} f(t)\,\mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t$$

comes from the neighbourhood of $t = 0$, because elsewhere the function is exponentially small. We can utilise this by splitting the integration interval in a convenient way as follows

$$= \int_0^{\sqrt{\varepsilon}} + \int_{\sqrt{\varepsilon}}^{\infty} f(t)\,\mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t = \varepsilon\int_0^{1/\sqrt{\varepsilon}} f(\varepsilon y)\,\mathrm{e}^{-y}\,\mathrm{d}y + \int_{\sqrt{\varepsilon}}^{\infty} f(t)\,\mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t$$

The last integral is exponentially small because

$$\left|\int_{\sqrt{\varepsilon}}^{\infty} f(t)\,\mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t\right| \leqslant \int_{\sqrt{\varepsilon}}^{\infty} |\,f(t)\,\mathrm{e}^{-t/\varepsilon}\,|\mathrm{d}t \leqslant \int_{\sqrt{\varepsilon}}^{\infty} K\,\mathrm{e}^{-t/\varepsilon}\,\mathrm{d}t = K\varepsilon\,\mathrm{e}^{-1/\sqrt{\varepsilon}}.$$

Since $f(\varepsilon y) = f(0) + \varepsilon y f'(0) + \frac{1}{2}\varepsilon^2 y^2 f''(0) + \ldots$ (uniformly) for $\varepsilon y \in [0, \sqrt{\varepsilon})$, we have finally

$$= \varepsilon\int_0^{1/\sqrt{\varepsilon}} \big(f(0) + \varepsilon y f'(0) + \ldots\big)\,\mathrm{e}^{-y}\,\mathrm{d}y + O(\varepsilon\,\mathrm{e}^{-1/\sqrt{\varepsilon}}) = \varepsilon f(0) + \varepsilon^2 f'(0) + \cdots + O(\varepsilon^{N+1}).$$

This result is a special case of

Theorem 8.2 (Watson's Lemma) *Let $f(t)$ be continuous*[3] *on $(0,\infty)$ and exponentially bounded, i.e. there is a real constant $c > 0$ such that $f(t) = O(\mathrm{e}^{ct})$ for $t \to \infty$. There are real constants $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ such that f has an asymptotic expansion for $t \to 0$ given by*

$$f(t) \sim \sum_{n=0}^{N} a_n t^{\lambda_n - 1} \quad \textit{for} \quad t \downarrow 0.$$

Then f's Laplace transform has the following asymptotic expansion for $s \to \infty$

$$F(s) = \int_0^{\infty} \mathrm{e}^{-st}\,f(t)\,\mathrm{d}t \sim \sum_{n=0}^{N} a_n \frac{\Gamma(\lambda_n)}{s^{\lambda_n}} \quad \textit{for} \quad s \to \infty \quad \textit{and} \quad |\arg(s)| \leqslant \beta \in (0, \tfrac{1}{2}\pi)$$

[2] A Laplace integral is an integral in the form of a Laplace transform.

[3] This can be relaxed, *e.g.* to piecewise continuous.

Proof
Since f is continuous on $(0,\infty)$ and $O(\mathrm{e}^{ct})$, there is for any $L>0$ a constant M with $|f(t)| \leqslant M\,\mathrm{e}^{ct}$ on $[L,\infty)$. Then for any s with $\mathrm{Re}(s)>c$ we have

$$\left|\int_L^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t\right| \leqslant \int_L^\infty \left|f(t)\,\mathrm{e}^{-st}\right|\,\mathrm{d}t \leqslant M\int_L^\infty \mathrm{e}^{-\mathrm{Re}(s)t+ct}\,\mathrm{d}t = \frac{M\,\mathrm{e}^{cL}}{\mathrm{Re}(s)-c}\,\mathrm{e}^{-\mathrm{Re}(s)L} = O(\mathrm{e}^{-sL}),$$

which is asymptotically for $s\to\infty$ smaller than any power of s. In a similar way is

$$\int_0^\infty t^{\lambda_n-1}\,\mathrm{e}^{-st}\,\mathrm{d}t = s^{-\lambda_n}\Gamma(\lambda_n) = \int_0^L t^{\lambda_n-1}\,\mathrm{e}^{-st}\,\mathrm{d}t + O(\mathrm{e}^{-sL}).$$

Choose a real constant K. Then (by assumption) there is an $L>0$ such that

$$\left|f(t)-\sum_{n=0}^N a_n t^{\lambda_n-1}\right| \leqslant K t^{\lambda_N-1} \quad \text{for} \quad 0<t<L.$$

Hence for s with $\mathrm{Re}(s)>0$

$$\left|\int_0^L\left(f(t)-\sum_{n=0}^N a_n t^{\lambda_n-1}\right)\mathrm{e}^{-st}\,\mathrm{d}t\right| \leqslant K\int_0^\infty t^{\lambda_N-1}\,\mathrm{e}^{-\mathrm{Re}(s)t}\,\mathrm{d}t = K\Gamma(\lambda_N)\,\mathrm{Re}(s)^{-\lambda_N} \leqslant K\Gamma(\lambda_N)|s^{-\lambda_N}|.$$

We write symbolically f for $f(t)\,\mathrm{e}^{-st}$ and Σ for $\sum_{n=0}^N a_n t^{\lambda_n-1}\,\mathrm{e}^{-st}$, and split

$$\int_0^\infty f\,\mathrm{d}t = \int_0^\infty (f-\Sigma)\,\mathrm{d}t + \int_0^\infty \Sigma\,\mathrm{d}t = \int_0^\infty \Sigma\,\mathrm{d}t + \int_0^L (f-\Sigma)\,dt + \int_L^\infty f\,dt - \int_L^\infty \Sigma\,dt.$$

After taking all parts together, we have, for $\mathrm{Re}(s)$ large enough (*i.e.* for large s inside a cone $|\arg(s)| \leqslant \beta\in[0,\frac{1}{2}\pi)$), the claimed result

$$F(s) = \sum_{n=0}^N a_n \frac{\Gamma(\lambda_n)}{s^{\lambda_n}} + o(s^{-\lambda_N}).$$

$\square$

In short, Watson's Lemma tells us when integration and asymptotic expansion in t can be exchanged, to result in an asymptotic expansion in s^{-1}.

Corollary 8.1 *Any finite integral* $\int_0^L f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$ *of a function of the form* $f(t)=t^\sigma g(t)$, *with* $\sigma>-1$ *and* $g(t)$ *analytic in* $t=0$, *satisfies the conditions of Watson's Lemma 8.2.*

Example 8.3

$$\int_0^\infty \frac{\mathrm{e}^{-st}}{t+1}\,\mathrm{d}t \sim \sum_{n=0}(-1)^n\frac{n!}{s^{n+1}}$$

$$\int_0^\infty \mathrm{e}^{-st}\log(1+t^2)\,\mathrm{d}t \sim \sum_{n=1}\frac{(-1)^{n+1}}{n}\frac{(2n)!}{s^{2n+1}}$$

$$\int_0^{\frac{1}{2}\pi}\mathrm{e}^{-s\tan^2(\theta)}\,\mathrm{d}\theta = \frac{1}{2}\int_0^\infty \frac{\mathrm{e}^{-st}}{\sqrt{t}(1+t)}\,\mathrm{d}t \sim \frac{1}{2}\sum_{n=0}(-1)^n\frac{\Gamma(n+\frac{1}{2})}{s^{n+\frac{1}{2}}}$$

Watson's Lemma is stronger than might appear at first sight. Many integrals can be recast by a coordinate transformation into the required form. More examples can be found in the exercises.

8.2 Laplace's Method

For generalisations of Laplace integrals, *i.e.* integrals of the form

$$f(s) = \int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,\mathrm{d}t$$

(with g and h sufficiently smooth) considered for $s \to \infty$, we may typically have the dominant contribution at an end of the interval or somewhere halfway.

Dominant contribution at left end. The dominant contribution is near $t = a$ if h is strict-monotonically increasing near a, *i.e.* $h'(a) > 0$, and h remains sufficiently bounded from below along the rest of the interval. Assume $g(a) \neq 0$. By a variant of Theorem 8.2 we have then

$$\int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,\mathrm{d}t \simeq \int_a^{a+s^{-\frac{1}{2}}} g(a)\,\mathrm{e}^{-sh(a)-s(t-a)h'(a)}\,\mathrm{d}t \simeq \frac{g(a)\,\mathrm{e}^{-sh(a)}}{s}\int_0^{s^{\frac{1}{2}}} \mathrm{e}^{-yh'(a)}\,\mathrm{d}t \simeq \frac{g(a)\,\mathrm{e}^{-sh(a)}}{sh'(a)}.$$

Dominant contribution at right end. The dominant contribution is near $t = b$ if h is strict-monotonically decreasing near b, *i.e.* $h'(b) < 0$, and h remains sufficiently bounded from below along the rest of the interval. In a similar way as before (assume $g(b) \neq 0$) we have then

$$\int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,\mathrm{d}t \simeq \int_{b-s^{-\frac{1}{2}}}^b g(b)\,\mathrm{e}^{-sh(b)-s(t-b)h'(b)}\,\mathrm{d}t \simeq \frac{g(b)\,\mathrm{e}^{-sh(b)}}{s}\int_{-s^{\frac{1}{2}}}^0 \mathrm{e}^{-yh'(b)}\,\mathrm{d}y \simeq -\frac{g(b)\,\mathrm{e}^{-sh(b)}}{sh'(b)}.$$

Dominant contribution halfway (Laplace's Method). Let h have an absolute minimum in $c \in (a,b)$, such that[4] $h'(c) = 0$, $h''(c) > 0$ and $h(t) = h(c) + \frac{1}{2}(t-c)^2 h''(c) + \ldots$. Let $g(c) \neq 0$ and h be sufficiently bounded from below along the rest of the interval. Then we obtain, by using $\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha t^2}\,\mathrm{d}t = \sqrt{\pi/\alpha}$, the asymptotic approximation for $s \to \infty$

$$\int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,\mathrm{d}t \simeq \int_{c-s^{-\frac{1}{4}}}^{c+s^{-\frac{1}{4}}} g(c)\,\mathrm{e}^{-sh(c)-\frac{1}{2}s(t-c)^2h''(c)}\,\mathrm{d}t \simeq \sqrt{\frac{2\pi}{sh''(c)}}\,g(c)\,\mathrm{e}^{-sh(c)}.$$

(Note: $O(s^{-\frac{1}{2}})$ here vs. $O(s^{-1})$ at the ends.) Proofs may be constructed in a similar way as with Watson's Lemma (8.2), by splitting the integration interval in an asymptotically small region near a, b, or c respectively for the dominant contribution, and a rest with a negligible contribution.

Example 8.4 The modified Bessel function $K_0(z)$ is for $z \to \infty$

$$K_0(z) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-z\cosh(t)}\,\mathrm{d}t \sim \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z},$$

by noting that $h(t) = \cosh(t)$ satisfies $h(0) = 1$, $h'(0) = 0$ and $h''(0) = 1$.

Example 8.5 A famous example is **Stirling's formula**. Via the transformation $t = n\tau$ is

$$n! = \Gamma(n+1) = \int_0^{\infty} t^n\,\mathrm{e}^{-t}\,\mathrm{d}t = n^{n+1}\int_0^{\infty} \mathrm{e}^{-n(\tau-\log\tau)}\,\mathrm{d}\tau \sim \sqrt{2\pi}\,n^{n+\frac{1}{2}}\,\mathrm{e}^{-n} \quad (n \to \infty),$$

by noting that $h(\tau) = \tau - \log(\tau)$ has $h(1) = 1$, $h'(1) = 0$ and $h''(1) = 1$.

[4] The case for vanishing higher derivatives is analogous.

8.3 Method of Stationary Phase

A generalisation of the above for Fourier-type integrals, *i.e.* integrals of the form

$$f(s) = \int_a^b g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}\, \mathrm{d}t$$

(with g absolutely integrable and h strictly monotonic except in discrete points) considered for $s \to \infty$, is similar, although there are differences. For example, proving the vanishing of the non-contributing parts of the integral takes more work. For this we need a version of the Riemann-Lebesgue Lemma including a generalisation.

Lemma 8.1 (Riemann-Lebesgue Lemma) *If the function $\phi(x)$ is absolutely integrable*[5]*, then the (finite or infinite) Fourier integral*

$$\int_a^b \phi(x)\, \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x \to 0 \quad as \quad |s| \to \infty.$$

Proof
A sketch of the proof is as follows. We write

$$\int_a^b \phi(x)\, \mathrm{e}^{\mathrm{i}sx}\, dx = -\int_a^b \phi\left(x + \frac{\pi}{s}\right) \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x + \int_a^{a+\frac{\pi}{s}} \phi(x)\, \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x - \int_b^{b+\frac{\pi}{s}} \phi(x)\, \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x$$

by a simple substitution, whence

$$\begin{aligned} 2\left|\int_a^b \phi(x)\, \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x\right| &= \left|\int_a^b \left\{\phi(x) - \phi\left(x + \frac{\pi}{s}\right)\right\} \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x + \int_a^{a+\frac{\pi}{s}} - \int_b^{b+\frac{\pi}{s}} \phi(x)\, \mathrm{e}^{\mathrm{i}sx}\, \mathrm{d}x\right| \\ &\leqslant \int_a^b \left|\phi(x) - \phi\left(x + \frac{\pi}{s}\right)\right| \mathrm{d}x + \int_a^{a+\frac{\pi}{s}} |\phi(x)|\, \mathrm{d}x + \int_b^{b+\frac{\pi}{s}} |\phi(x)|\, \mathrm{d}x \end{aligned}$$

which tend to 0 as $|s| \to \infty$ by fundamental theorems of integration. □

Corollary 8.2 *If the function $\mu(x)$ is strictly monotonic, with $|\mu'(x)| > \delta > 0$, such that we can define the inverse $x = \mu^{-1}(z)$, then*

$$\int_a^b \phi(x)\, \mathrm{e}^{\mathrm{i}s\mu(x)}\, \mathrm{d}x = \int_{\mu(a)}^{\mu(b)} \frac{\phi(\mu^{-1}(z))}{\mu'(\mu^{-1}(z))}\, \mathrm{e}^{\mathrm{i}sz}\, \mathrm{d}z \to 0 \quad as \quad |s| \to \infty.$$

If we return to the original integral, we may typically have the dominant contribution at the ends of the interval or somewhere halfway.

Dominant contribution at the ends. The dominant contribution is at the ends if h is strictly monotonic, such that we can write by partial integration

$$f(s) = \int_a^b g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}\, \mathrm{d}t = \frac{g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}}{\mathrm{i}sh'(t)}\bigg|_a^b - \frac{1}{\mathrm{i}s}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{g(t)}{h'(t)}\right] \mathrm{e}^{\mathrm{i}sh(t)}\, \mathrm{d}t \simeq \frac{g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}}{\mathrm{i}sh'(t)}\bigg|_a^b$$

since by the Riemann-Lebesgue Lemma the second integral decays faster than $O(1/s)$.

[5] This condition can be relaxed to piecewise continuous and uniform convergence at the ends and discontinuities.

Dominant contribution halfway (Method of Stationary Phase). Let h have a 2-nd order stationary point in $c \in (a,b)$, such that $h'(c)=0$, $h''(c) \neq 0$ and $h(t) = h(c) + \frac{1}{2}(t-c)^2 h''(c) + \dots$, while h is sufficiently smooth and strictly monotonic along the rest of the interval, and $g(c) \neq 0$. Then we obtain, by using $\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}\alpha t^2}\,\mathrm{d}t = \mathrm{e}^{\pm\frac{1}{4}\pi\mathrm{i}}\sqrt{\pi/\alpha}$, the asymptotic approximation for $s \to \infty$

$$\int_a^b g(t)\,\mathrm{e}^{\mathrm{i}sh(t)}\,\mathrm{d}t \simeq \int_{c-s^{-\frac{1}{4}}}^{c+s^{-\frac{1}{4}}} g(c)\,\mathrm{e}^{\mathrm{i}sh(c)+\frac{1}{2}\mathrm{i}s(t-c)^2h''(c)}\,\mathrm{d}t \simeq g(c)\,\mathrm{e}^{\mathrm{i}sh(c)}\,\mathrm{e}^{\pm\frac{1}{4}\pi\mathrm{i}}\sqrt{\pm\frac{2\pi}{sh''(c)}}\,,$$

where the $\pm$ sign can be chosen according to what is most convenient. Similar to Laplace's Method, a contribution here is $O(s^{-\frac{1}{2}})$, while $O(s^{-1})$ at the ends. The limit $s \to -\infty$, and stationary points of higher order can be dealt with in an analogous way (make sure to distinguish even and odd orders). Proofs may be found by splitting the integration interval in an asymptotically small region near c, and a remaining part where the Riemann-Lebesgue Lemma can be applied.

Example 8.6 The n-th order Bessel function of the first kind $J_n(x)$ (section 10.1) has an asymptotic behaviour for large values of the argument x, given by

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\,\mathrm{Re}\Big[\int_0^{\pi} \mathrm{e}^{\mathrm{i}x\sin t - \mathrm{i}nt}\,\mathrm{d}t\Big] \\ &\simeq \frac{1}{\pi}\,\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}n\frac{1}{2}\pi}\,\mathrm{e}^{\mathrm{i}x\sin(\frac{1}{2}\pi)}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\sqrt{\frac{2\pi}{x\sin(\frac{1}{2}\pi)}}\,\right] = \sqrt{\frac{2}{x\pi}}\cos\!\big(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\big) \end{aligned}$$

due to the stationary point at $t = \frac{1}{2}\pi$.

Example 8.7 The asymptotic behaviour of $J_n(n)$ for large n cannot be found by the standard formula. However, a slight adaptation following the same lines of reasoning yields

$$J_n(n) = \frac{1}{\pi}\,\mathrm{Re}\Big[\int_0^{\pi} \mathrm{e}^{\mathrm{i}n(\sin t - t)}\,\mathrm{d}t\Big] \simeq \frac{1}{\pi}\,\mathrm{Re}\Big[\int_0^{\infty} \mathrm{e}^{-\mathrm{i}\frac{1}{6}nt^3}\,\mathrm{d}t\Big] = \frac{2^{\frac{1}{3}}}{n^{\frac{1}{3}}3^{\frac{2}{3}}\Gamma(\frac{2}{3})}$$

due to phase function $h(t) = \sin(t) - t = -\frac{1}{6}t^3 + \dots$, expanded around the stationary point $t = 0$.

Example 8.8 Another example that requires some preparation is the integral

$$\int_0^{\frac{1}{2}\pi} t\sin(x\cos t)\,\mathrm{d}t = \mathrm{Im}\Big[\int_0^{\frac{1}{2}\pi} t\,\mathrm{e}^{\mathrm{i}x\cos t}\,\mathrm{d}t\Big]$$

The point of stationary phase is at left end $t = 0$, which would normally lead to half its contribution, but at the same time the function $g(t) = t$ vanishes there. However, by partial integration we can compensate for this and find the dominating contributions from the ends

$$= \mathrm{Im}\Big[\frac{\mathrm{i}t}{x\sin t}\,\mathrm{e}^{\mathrm{i}x\cos t}\Big|_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{i}t}{x\sin t}\Big)\mathrm{e}^{\mathrm{i}x\cos t}\,\mathrm{d}t\Big] = \frac{1}{x}\big(\tfrac{1}{2}\pi - \cos x\big) + o(x^{-1}).$$

8.4 Method of Steepest Descent or Saddle Point Method

The Method of Steepest Descent (Peter Debije, 1909) is essentially a recipe to optimise the integration contour of an integral (for analytic functions g and h) of the form

$$f(s) = \int_{\mathcal{C}} g(z)\,\mathrm{e}^{-sh(z)}\,\mathrm{d}z,$$

such that the integral is, after parametrisation, amenable to Laplace's method.

As will be shown, the contribution to the integral comes primarily from the neighbourhood of the point (or points) where $h'(z) = 0$. Write $h = u + \mathrm{i}v$. Suppose h' vanishes at a point z_0 with $h(z_0) = u_0 + \mathrm{i}v_0$. (We start with the assumption that $h''(z_0) \neq 0$, but higher order generalisations are similar.) In order to make the integral not oscillatory anymore, we deform the contour $\mathcal{C}$ (at least for the part where h dominates) into a contour $\mathcal{L}$ where $v = v_0$ is constant. Such a contour is at the same time a contour of **steepest descent** for u: (i) ∇v is orthogonal to level curve $\mathcal{L}$; (ii) because of the Cauchy-Riemann relations is ∇u, the direction where u varies most, orthogonal to ∇v, and hence directed along $\mathcal{L}$.

Since $h'(z_0) = 0$, $h(z) = h(z_0) + \frac{1}{2}h''(z_0)(z - z_0)^2 + \ldots$. If we write $\frac{1}{2}h''(z_0) = \rho\,\mathrm{e}^{\mathrm{i}\alpha}$ and $z = z_0 + r\,\mathrm{e}^{\mathrm{i}\theta}$, it follows that $u = u_0 + \rho r^2 \cos(2\theta + \alpha)$, from which can be concluded that z_0 is a **saddle point** of u. Therefore, there are two steepest descent contours $v = v_0$: one of hill-type and one of valley-type, which intersect at z_0. Obviously, only the valley-type is useful, where u has a minimum at $z = z_0$. Along $\mathcal{L}$ we have thus $h = u + \mathrm{i}v_0$ where u is real. With parametrisation $z = \gamma(t)$, with $z_0 = \gamma(0)$, is then $h(\gamma(t)) = h(z_0) + \frac{1}{2}\beta t^2 + \ldots$ where $\beta = h''(z_0)\gamma'(0)^2$ is by the above construction real positive. Taking note of the direction z_0 is crossed, we obtain

$$f(s) = \int_{\mathcal{L}} g(z)\,\mathrm{e}^{-sh(z)}\,\mathrm{d}z = \int_a^b g(\gamma(t))\,\mathrm{e}^{-sh(\gamma(t))}\,\gamma'(t)\,\mathrm{d}t \simeq \pm\sqrt{\frac{2\pi}{sh''(z_0)}}g(z_0)\,\mathrm{e}^{-sh(z_0)}\,.$$

The method is best explained by an example:

Example 8.9 A classic example is Hankel's asymptotic expansion of the Bessel functions. Consider (see section 10.1) the representation of the n-th order Hankel function of the first kind

$$H_n^{(1)}(s) = \frac{1}{\pi\mathrm{i}}\int_{\mathcal{C}} \mathrm{e}^{s\sinh z - nz}\,\mathrm{d}z,$$

with integration contour $\mathcal{C}$ from $-\infty$ to $\infty + \pi\mathrm{i}$. We are interested in its behaviour for $s \to \infty$.

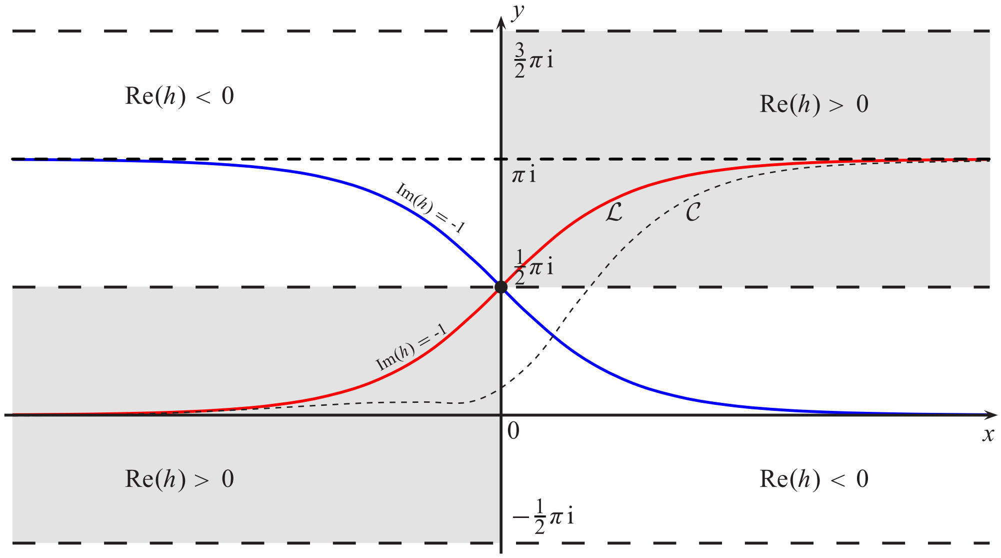
Consider the landscape of the following function

$$h(z) = -\sinh(z) = -\sinh(x)\cos(y) - \mathrm{i}\cosh(x)\sin(y).$$

$h(z)$ has a stationary point, where $h'(z) = 0$, at $z_0 = \frac{1}{2}\pi\mathrm{i}$. It is clearly a saddle point of $\mathrm{Re}(h)$, as $\mathrm{Re}(h)$ is negative in the right-lower and left-upper semi-infinite strip (see the figure below), and positive in the right-upper and left-lower strip (gray). Evidently, the integration contour $\mathcal{C}$ has to run from the left-lower to the right-upper strip, otherwise the integral would not converge. There are two paths of steepest descent of $\mathrm{Re}(h)$ through saddle point z_0. They are given by $\mathrm{Im}(h) = -\cosh(x)\sin(y) = -1$. After some algebra they are found to be given by

$$y = 2\arctan(\mathrm{e}^{x}) \quad \text{(red)}, \qquad y = 2\arctan(\mathrm{e}^{-x}) \quad \text{(blue)}.$$

For the blue path, z_0 is a maximum of $\mathrm{Re}(h)$, and this path is useless for our purposes. For the red path, however, it is a minimum. So we deform $\mathcal{C}$ into the red path $\mathcal{L}$, in order to be able to apply Laplace's method.



With the following parametrisation γ and its properties

$$z = \gamma(t) = t + \mathrm{i}\,2\arctan(\mathrm{e}^t), \quad \gamma'(t) = 1 + \mathrm{i}\cosh(t)^{-1}, \quad \sinh\gamma(t) = -\sinh(t)\tanh(t) + \mathrm{i}$$

we obtain

$$H_n^{(1)}(s) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-s\gamma(t) - n\gamma(t)}\,\gamma'(t)\,\mathrm{d}t = \frac{\mathrm{e}^{\mathrm{i}s}}{\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-s\sinh(t)\tanh(t) - nt - \mathrm{i}2n\arctan(\mathrm{e}^t)}(1 + \mathrm{i}\cosh(t)^{-1})\,\mathrm{d}t,$$

Since the contribution of the integrand[6] is now concentrated near $t = 0$ (*i.e.* z_0), we have

$$H_n^{(1)}(s) \simeq \frac{\mathrm{e}^{\mathrm{i}s}}{\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-st^2 - \mathrm{i}2n\arctan(1)}(1+\mathrm{i})\,\mathrm{d}t = \sqrt{\frac{2}{\pi s}}\,\mathrm{e}^{\mathrm{i}s - \frac{1}{2}\pi n\mathrm{i} - \frac{1}{4}\pi\mathrm{i}} \qquad (s \to \infty).$$

[6] For $n \gtrsim \sqrt{s}$ and $st^2 = O(1)$, the term nt is not negligible against $-st^2$ and the approximation breaks down.

8.5 Applications

8.5.1 Group velocity

Many linear waves in x and t, for example water waves, are such that components $\sim \mathrm{e}^{-\mathrm{i}kx}$ of wave number k only exist in combination with components $\sim \mathrm{e}^{\mathrm{i}\omega t}$ of a specific frequency ω. This frequency is given by a relation $\omega = \Omega(k)$, called the dispersion relation. By superposition these waves can be written in general by the Fourier integral

$$\psi(x,t) = \int_{-\infty}^{\infty} g(k)\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kx}\,\mathrm{d}k = \int_{-\infty}^{\infty} g(k)\,\mathrm{e}^{\mathrm{i}t(\Omega(k)-k\frac{x}{t})}\,\mathrm{d}k.$$

Applying the method of stationary phase we conclude that an observer, moving with velocity v along $x = vt$, sees for large t only the component of wave number k_v given by the stationary phase

$$\frac{\mathrm{d}}{\mathrm{d}k}\big(\Omega(k) - kv\big) = \Omega'(k) - v = 0.$$

In other words, these waves propagate with the group velocity, $c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \Omega'(k)$, rather than the phase velocity, $c_f = \frac{\omega}{k} = \Omega(k)/k$. With $\omega_v = \Omega(k_v)$ we have eventually

$$\psi(x,t) \simeq g(k_v)\,\mathrm{e}^{\mathrm{i}\omega_v t - \mathrm{i}k_v x}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\sqrt{\frac{2\pi}{t\Omega''(k_v)}} \quad \text{for} \quad t \to \infty \quad \text{along} \quad x = vt.$$

Linear water waves on depth h with gravity g and neglecting surface tension satisfy $\omega = \Omega(k) = \sqrt{gk\tanh(kh)}$, which is for deep water $\simeq \sqrt{gk}$. The group velocity $c_g = \frac{1}{2}\sqrt{g/k}$ is then exactly half the phase velocity $c_f = \sqrt{g/k}$. This is nicely seen when throwing a stone in a pond.

8.5.2 Doppler effect of a moving sound source.

The observed pitch of a moving sound source of frequency ω_0 is higher if the source approaches the observer and lower if it recedes from it. This frequency shift, called *the Doppler effect*, occurs if the time scale of the tone ω_0^{-1} is much smaller than the time scale T of the motion, *i.e.* if $\omega_0 T \gg 1$.

Let the sound field $p(\boldsymbol{x},t)$ of a time-harmonic point source, of frequency ω_0 and moving subsonically along the path $\boldsymbol{x} = \boldsymbol{x}_S(t)$, be given by the following inhomogeneous wave equation

$$c_0^{-2}\tfrac{\partial^2}{\partial t^2}p - \nabla^2 p = 4\pi q_0\,\mathrm{e}^{\mathrm{i}\omega_0 t}\,\delta(\boldsymbol{x} - \boldsymbol{x}_S(t)).$$

According to Liénard and Wiechert[7] the solution in free space is given by

$$p(\boldsymbol{x},t) = \frac{q_0\,\mathrm{e}^{\mathrm{i}\omega_0 t_e}}{R_e(1 - M_e\cos\theta_e)}$$

where $t_e = t_e(\boldsymbol{x},t)$ is the emission time. This is the time of emission of the signal that travelled (along a straight line with the sound speed c_0) from the source in $\boldsymbol{x}_S$ at time t_e to the observer in $\boldsymbol{x}$ at time t. It is a function of $\boldsymbol{x}$ and t, implicitly given by the equation

$$t = t_e + \big\|\boldsymbol{x} - \boldsymbol{x}_S(t_e)\big\|c_0^{-1}.$$

[7] The so-called **Liénard-Wiechert potential** is independently found in 1898 and 1900.

For subsonically moving sources, this equation has exactly one solution. Furthermore, the distance R_e between source and observer, the Mach number M_e of the source speed, and the angle θ_e between the observer direction and the source velocity, are functions of t_e and given by

$$R_e = \left\|\boldsymbol{x} - \boldsymbol{x}_S(t_e)\right\|, \quad M_e = \frac{\left\|\dot{\boldsymbol{x}}_S(t_e)\right\|}{c_0}, \quad \cos\theta_e = \frac{(\boldsymbol{x} - \boldsymbol{x}_S(t_e)) \cdot \dot{\boldsymbol{x}}_S(t_e)}{\left\|\boldsymbol{x} - \boldsymbol{x}_S(t_e)\right\| \left\|\dot{\boldsymbol{x}}_S(t_e)\right\|}.$$

Let the typical time associated to the source motion be T, so we can write $\boldsymbol{x}_S(t) = X_0 \boldsymbol{\xi}(t/T)$, where X_0 characterises a typical position and $\boldsymbol{\xi} = O(1)$ is a dimensionless function. Assuming that time variations due to frequency ω_0 are much larger than T, we are interested in the Fourier-transformation in time of p.

For this we ignore for the moment the $\boldsymbol{x}$ dependence and consider for smoothly varying amplitude A and phase $\omega_0 T\phi$ the slowly varying, *almost* harmonic signal p

$$p(t) = A(t/T)\, \mathrm{e}^{\mathrm{i}\omega_0 T\phi(t/T)}, \quad \omega_0 T \gg 1.$$

If T is large compared to t, then $\mathrm{e}^{\mathrm{i}\omega_0 T\phi(t/T)} = \mathrm{e}^{\mathrm{i}\omega_0 T\phi(0) + \mathrm{i}\omega_0\phi'(0)t + \cdots}$ and we see that the observed frequency is about $\omega_0\phi'(0)$. To generalise this for arbitrary t, we consider the Fourier transform

$$P(\omega) = \int_{-\infty}^{\infty} p(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t,$$

to see which part of the spectrum dominates and when. We assume that ω is of the order of magnitude of ω_0. Introduce the large parameter $\lambda = \omega_0 T$. Make time t dimensionless on the slow time scale, with $t = T\tau$. Scale ω on ω_0 such that $\omega = \omega_0\nu$, with $\nu = O(1)$. We obtain then

$$P(\omega) = P(\omega_0\nu) = T\int_{-\infty}^{\infty} A(\tau)\, \mathrm{e}^{\mathrm{i}\lambda\phi(\tau) - \mathrm{i}\lambda\nu\tau}\, \mathrm{d}\tau,$$

For large λ this becomes, by using the method of stationary phase,

$$P(\omega_0\nu) \simeq T A(\tau_s)\, \mathrm{e}^{\mathrm{i}\lambda\phi(\tau_s) - \mathrm{i}\lambda\nu\tau_s}\, \mathrm{e}^{\frac{1}{4}\pi\mathrm{i}} \sqrt{\frac{2\pi}{\lambda\phi''(\tau_s)}}$$

with $\tau_s = \tau_s(\nu)$ defined by the stationary phase equation

$$\phi'(\tau_s) = \nu.$$

In other words: to find frequency $\omega = \omega_0\nu = \omega_0\phi'(t_s/T)$ we have to look at time $t = t_s = T\tau_s$. Therefore, $\omega_0\phi'(t/T)$ is sometimes called the *instantaneous frequency*.

Returning to our original problem of the moving point source with $t_e = T\phi(t/T)$, we find that the instantaneous frequency ω observed at position $\boldsymbol{x}$ and time t is then given by[8]

$$\omega = \omega_0\phi'(t/T) = \omega_0\frac{\mathrm{d}t_e}{\mathrm{d}t} = \frac{\omega_0}{1 - M_e\cos\theta_e}.$$

This formula expresses the famous Doppler shift.

[8] Check. It requires implicit differentiation.

8.6 Integral Asymptotics: Assignments

8.6.1 Integrals and Watson's Lemma

1. Let ϕ be a smooth function (continuously differentiable) on $[0, a]$. Prove that for $\varepsilon \to 0$

   (a)
   $$\int_0^a \frac{\phi(t)}{t+\varepsilon}\,\mathrm{d}t = -\phi(0)\log\varepsilon + O(1)$$

   (b)
   $$\int_0^a \frac{\phi(t)}{t^2+\varepsilon^2}\,\mathrm{d}t = \frac{\pi}{2\varepsilon}\phi(0) + O(\log\varepsilon)$$

   HINT. Write $\phi(t) = \phi(t) - \phi(0) + \phi(0)$ and prove that $\exists(K > 0)$ with $|\phi(t) - \phi(0)| \leqslant Kt$.

2. Find the (leading order) asymptotic behaviour for $x \to \infty$ of
   $$\int_0^\infty \frac{\log(1+1/t)}{t+x}\,\mathrm{d}t.$$
   HINT. Split the integration interval in $[0, \lambda] \cup [\lambda, \infty)$ with $\lambda = O_s(\sqrt{x})$.

3. Find, by applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of
   $$\int_0^\infty \mathrm{e}^{-xt}\sin t\,\mathrm{d}t.$$

4. Find, by introducing the new variable $y = t^3$ and applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of
   $$\int_0^\infty \frac{\mathrm{e}^{-xt^3}}{1+t}\,\mathrm{d}t.$$

5. Find, by introducing the new variable $y = (\sin t)^4$ and applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of
   $$\int_0^{\frac{1}{2}\pi} \sqrt{\sin t}\,\mathrm{e}^{-x(\sin t)^4}\,\mathrm{d}t.$$

6. Find, by introducing $t = x(y+1)$ and applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of the exponential integral (c.f. section 10.3, and example 8.3)
   $$E_1(x) = \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

7. Let $f(t)$ be continuous on $[0, \infty)$, analytic in $t = 0$ and bounded for $t \to \infty$. Let $\mu > 0$. Find the asymptotic behaviour for $z \to \infty$ of
   $$F(z) = \int_0^\infty f(t)\,\mathrm{e}^{-zt^{1/\mu}}\,\mathrm{d}t.$$
   HINT. Transform $t = y^\mu$ and apply Watson's Lemma.

8. A representation of the Airy function $\mathrm{Ai}(x)$ is given by

$$\mathrm{Ai}(x) = \frac{\mathrm{e}^{-\frac{2}{3}x^{\frac{3}{2}}}}{\pi} \int_0^\infty \mathrm{e}^{-x^{\frac{1}{2}}t^2} \cos\left(\tfrac{1}{3}t^3\right) \mathrm{d}t.$$

   Find its asymptotic behaviour for $x \to \infty$. HINT. Transform $t^2 = \tau$.

9. Find the asymptotic behaviour for $m \to \infty$ of

$$S(m) = \int_0^\pi \left(\frac{\sin t}{t}\right)^m \mathrm{d}t$$

   by introducing $\tau = \ln(t/\sin t)$ and showing that

$$S(m) = \int_0^\infty \mathrm{e}^{-m\tau} \frac{\mathrm{d}t}{\mathrm{d}\tau} \mathrm{d}\tau, \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{t \sin t}{\sin t - t \cos t} = \sqrt{\frac{3}{2\tau}} + \dots \quad (\tau \to 0).$$

10. Generalise Watson's Lemma for integrals of the form

$$\int_0^\infty t^{\mu-1} \ln(t) f(t) \mathrm{e}^{-xt} \mathrm{d}t$$

   where f is analytic in $t = 0$. You may use the identity

$$\int_0^\infty t^{x-1} \ln(t) \mathrm{e}^{-t} \mathrm{d}t = \Gamma'(x) = \psi(x)\Gamma(x).$$

11. Find an asymptotic expansion for $\varepsilon \to 0$ of the integral

$$\int_0^\infty \mathrm{e}^{-t} t^\mu f(\varepsilon t) \mathrm{d}t$$

   where $\mu > -1$, and $f(x)$ is analytic in $x = 0$ and exponentially bounded for $x \to \infty$.

12. Find an asymptotic expansion for $\varepsilon \to 0$ of the integral

$$\int_0^\infty \frac{\mathrm{e}^{-t}}{(1+\varepsilon t)^3} \mathrm{d}t.$$

13. Consider for $x, a, s > 0$ the function $h(x; a, s) = \mathrm{e}^{-(a/x)^s}$. Show that for $s \to \infty$, a fixed and $x \in (0, \infty)$ it tends to the unit step function $H(x-a)$, by proving that for a smooth and integrable testfunction $\phi(x)$ the following integral satisfies

$$\int_0^\infty h(x; a, s)\phi(x) \mathrm{d}x = \int_0^\infty H(x-a)\phi(x) \mathrm{d}x - s^{-1}\gamma\phi(a) + o(s^{-1})$$

   You may assume that there is some number $K > a$ such that $\phi(x) = 0$ for $x > K$.

8.6.2 Laplace's Method

1. (a) Find, by introducing $t = x\sqrt{y+1}$ and applying Watson's Lemma,
   (b) Find, by introducing $t = x + y$ and applying Watson's Lemma,
   (c) Find, by introducing $t = xy$ and applying a version of Laplace's method,

   the asymptotic behaviour for $x \to \infty$ of the complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\, \mathrm{d}t.$$

   Compare with the exact values of $\operatorname{erfc}(2)$ and $\operatorname{erfc}(4)$.

2. Find the asymptotic behaviour for $x \to \infty$ of the real function $K(x)$, defined for $x > 0$ by

$$K(x) = \int_{-\infty}^{\infty} \mathrm{e}^t\, \mathrm{e}^{-xh(t)}\, \mathrm{d}t, \quad \text{where} \quad h(t) = \mathrm{e}^t - t.$$

3. Consider, for real a, b, x, and real and smooth g, h, the following integral

$$\int_a^b \mathrm{e}^{-xh(t)}\, g(t)\, \mathrm{d}t$$

   asymptotically for $x \to \infty$. Let $h(t)$ attain its minimum at the interior point $t_0 \in (a, b)$, while $h''(t_0) > 0$. Show that the first term of the asymptotic expansion is

$$g(t_0)\, \mathrm{e}^{-xh(t_0)} \sqrt{\frac{2\pi}{xh''(t_0)}}\,.$$

   Write out the next few terms.

4. Find the asymptotic behaviour for $x \to \infty$ of

$$\int_0^1 t^x \sin(t)^2\, \mathrm{d}t$$

5. Find, with $\alpha > 0$, the asymptotic behaviour for $x \to \infty$ of

$$\int_0^\infty \mathrm{e}^{-xt + x^\alpha \log(t)}\, \mathrm{d}t$$

   HINT. Transform $t = x^{\alpha-1} y$ and apply Laplace's method.

6. Verify that if in Laplace's Method $g(c) = 0$, the main contribution is from $g''(c)$, provided this is nonzero. Find the asymptotic behaviour for large x of the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-xt^2} \ln(1 + t + t^2)\mathrm{d}t.$$

8.6.3 Method of Stationary Phase

1. Find, by using the Method of Stationary Phase, the asymptotic behaviour of the n-th order Bessel function $J_n(x)$ for $x, n \to \infty$ at fixed ratio n/x. See example 8.6. Assume $n/x < 1$.

2. Find, by introducing $t = \sqrt{x}y$, the asymptotic behaviour for large x of the Airy function

$$\mathrm{Ai}(-x) = \frac{1}{\pi}\int_0^\infty \cos(xt - \tfrac{1}{3}t^3)\mathrm{d}t.$$

3. a) Show that, as $x \to \infty$,

$$\int_0^\pi f(t)\,\mathrm{e}^{\mathrm{i}x\psi(t)}\,\mathrm{d}t \simeq f(0)\,\mathrm{e}^{\mathrm{i}xa+\frac{1}{6}\pi\mathrm{i}}\Big(\frac{1}{27bx}\Big)^{\frac{1}{3}}\Gamma(\tfrac{1}{3})$$

where $f(t)$ and $\psi(t)$ are smooth, $\psi'(t) \neq 0$ for $t > 0$, $f(0) \neq 0$, $\psi(0) = a$, $\psi'(0) = \psi''(0) = 0$ and $\psi'''(0) = 6b > 0$.

b) Consider the Bessel function (section 10.1)

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\,\mathrm{d}t.$$

Show that, as $n \to \infty$,

$$J_n(n) \simeq \frac{\Gamma(\frac{1}{3})}{\pi(48)^{\frac{1}{6}}n^{\frac{1}{3}}}$$

Mathematical Tripos Part II, 2013

4. a) Describe how the leading-order asymptotic behaviour as $x \to \infty$ of

$$I(x) = \int_a^b f(t)\,\mathrm{e}^{\mathrm{i}xg(t)}\,\mathrm{d}t$$

may be found by the method of stationary phase, where f and g are real functions and the integral is taken along the real line. You should consider the cases for which:

(i) $g'(t)$ is non-zero in $[a, b)$ and has a simple zero at $t = b$.

(ii) $g'(t)$ is non-zero apart from having one simple zero at $t = t_0$, where $a < t_0 < b$.

(iii) $g'(t)$ has more than one simple zero in (a, b) with $g'(a) \neq 0$ and $g'(b) \neq 0$.

b) Use the method of stationary phase to find the leading-order asymptotic form as $x \to \infty$ of

$$J(x) = \int_0^1 \cos\big(x(t^4 - t^2)\big)\,\mathrm{d}t.$$

Mathematical Tripos Part II, 2014

8.6.4 Method of Steepest Descent or Saddle Point Method

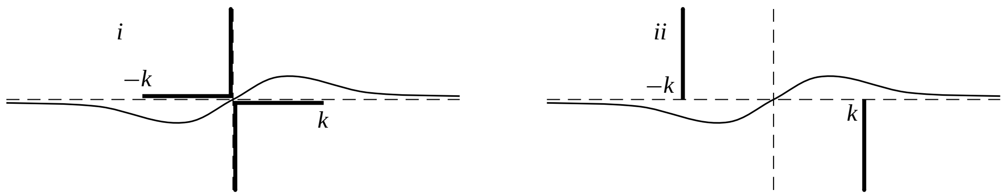
1. By Fourier transformation to x of the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+k^2\phi=0$$

(where $k>0$), solutions in $x\in(-\infty,\infty)$, $y\in[0,\infty)$, can be found of the form

$$\phi(x,y)=\int_{-\infty}^{\infty}F(\alpha)\,\mathrm{e}^{-\mathrm{i}\alpha x-\mathrm{i}\gamma y}\,\mathrm{d}\alpha,\quad \gamma(\alpha)=\sqrt{k^2-\alpha^2},\quad F(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_0(x)\,\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x,$$

for given $\phi_0(x)=\phi(x,0)$. In view of conditions of boundedness for $y\to\infty$ we choose[9] a branch and branch cuts of γ such that $\mathrm{Im}(\gamma)\leqslant 0$ and $\gamma(0)=k$. An integration contour is taken that never crosses the branch cuts (figure i).



Since we want to deform the contour into a steepest descent contour, it is more convenient to rotate the branch cuts[10] away as in figure ii. The integrand remains the same along the contour, so the integral does not change. We transform $\alpha=ku$, introduce $w(u)=\sqrt{1-u^2}$ analogous to γ, and rewrite

$$\phi(x,y)=k\int_{-\infty}^{\infty}F(ku)\,\mathrm{e}^{-krh(u)}\,\mathrm{d}u,$$

where

$$h(u)=\mathrm{i}u\cos\theta+\mathrm{i}w(u)\sin\theta,\quad x=r\cos\theta,\quad y=r\sin\theta,\quad 0\leqslant\theta\leqslant\pi.$$

Let F be given. Consider ϕ asymptotically for large kr.

(a) Determine the asymptotic behaviour of $\phi(x,y)$ for $kr\to\infty$ by application of the Method of Stationary Phase, with the integration contour taken along the real axis.

(b) Find the saddle point u_0 of the phase function $\mathrm{Re}\,h(u)$. Note the sign of w.

(c) Determine the contour of steepest descent through u_0 for which u_0 is a minimum.
HINT. Consider $h(u)=h(u_0)+\lambda^2$ and solve for $u=u(\lambda)$. Make a plot.

(d) Determine the asymptotic behaviour of $\phi(x,y)$ for $kr\to\infty$ by application of the Steepest Descent Method. Compare your result with the one found in a).

(e) The far field radiation pattern $D(\theta)=\lim_{kr\to\infty}\sqrt{kr}\,|\phi(x,y)|$ is independent of r and signifies the angular dependence of the radiated field strength $|\phi|$.
Determine $D(\theta)$ for a source of the form $\phi_0(x)=\mathrm{e}^{-\mathrm{i}\alpha_0 x-\frac{1}{2}(x/L)^2}$, $\alpha_0=\pm k\cos\theta_0$. (First, determine $F(\alpha)$ from $\phi_0(x)$.) What happens if kL becomes large enough?

[9] Expressed in principal value square roots, this is $\gamma(\alpha)=-\,\mathrm{sign}\big(\mathrm{Im}\big(\sqrt{k^2-\alpha^2}\,\big)\big)\sqrt{k^2-\alpha^2}$ with $\gamma(+0+\mathrm{i}0)=k$.
[10] Expressed in principal value square roots, this is $\gamma(\alpha)=-\mathrm{i}\sqrt{\mathrm{i}(k-\alpha)}\,\sqrt{\mathrm{i}(k+\alpha)}$ with $\gamma(0)=k$.

2. Consider the integral

$$I(x) = \int_0^1 \frac{1}{\sqrt{t - t^2}} \, \mathrm{e}^{\mathrm{i}x(t^2+t)} \, \mathrm{d}t$$

for real $x > 0$. Find and sketch, in the complex t-plane, the paths of steepest descent through the endpoints $t = 0$ and $t = 1$ and through any saddle point(s). Obtain the leading order term in the asymptotic expansion of $I(x)$ for large positive x.

Mathematical Tripos Part II, 2016

Chapter 9

Some Mathematical Auxiliaries

9.1 Phase plane

Phase portrait and phase plots

A differentiable function $\phi(t)$, defined on some (not necessarily finite) interval $t \in [a, b]$, can be portrayed by the parametric curve (x, y) in $\mathbb{R}^2$, where $x = \phi(t)$ and $y = \phi'(t)$. This curve is called a phase portrait or phase plot of ϕ, and the (ϕ, ϕ')-plane is called a phase plane.

Phase plots are particularly useful if ϕ is defined by a differential equation from which relations between ϕ and ϕ' can be obtained, but exact solutions are not or not easily found.

Important examples are

$$\phi(t) = A\cos(\omega t), \quad \phi'(t) = -\omega A\sin(\omega t), \quad \text{with} \quad \omega^2\phi^2 + \phi'^2 = \omega^2 A^2,$$

leading to an ellipse as phase plot. A variant is

$$\phi(t) = A\,\mathrm{e}^{-ct}\cos(\omega t), \quad \phi'(t) = \sqrt{(\omega^2 + c^2)}\,A\,\mathrm{e}^{-ct}\sin(\omega t - \arctan(\omega/c) - \tfrac{1}{2}\pi)$$

leading to an elliptic spiral, converging to the origin if $c > 0$ and diverging to infinity if $c < 0$.

Phase plot to illustrate the solutions of differential equation

A differential equation like the harmonic equation

$$y'' + \omega^2 y = 0$$

is simple enough to be solved exactly by $y(t) = A\cos(\omega t) + B\sin(\omega t)$, leading to periodic (circular or elliptic) phase plots (see above). More difficult, in particular nonlinear, differential equations cannot be solved exactly, and solutions have to be found (in general) numerically. The plot of a single solution, however, does not tell us much about the whole family of all possible solutions. In such a case it is instructive to create a phase plot. Take for example the Van der Pol equation

$$y'' + y - \varepsilon(1 - y^2)y' = 0.$$

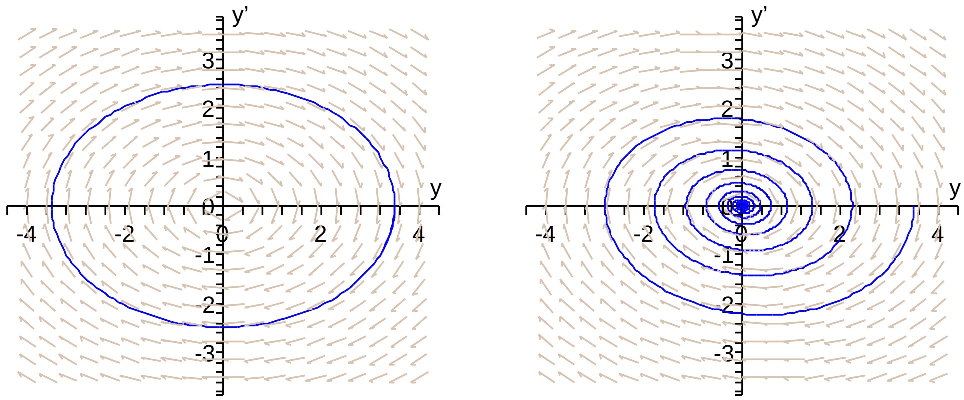


Figure 9.1: Elliptic (periodic) and spiral (damped) phase plots.

For small enough $\|(y, y')\|$, the nonlinear term is on average negative and acts as a source leading to an increase. For large enough $\|(y, y')\|$, the nonlinear term is on average positive and acts as a sink leading to a decay. From outside inwards and from inside outwards, these solutions converge to a periodic solution with (for small ε) an amplitude of about 2.

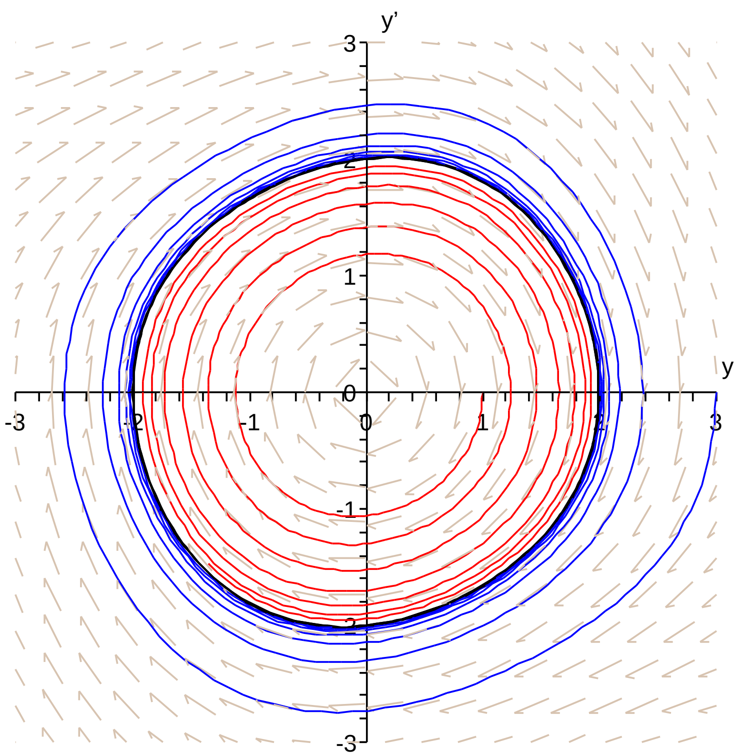


Figure 9.2: A phase plot of the Van der Pol equation, with $\varepsilon = 0.1$ and solutions starting form $y' = 0$ with $y = 1$ (red) and $y = 3$ (blue), respectively.

Stability of stationary solutions

One of the most important applications of the phase plot is the stability analysis of stationary solutions of 2nd order autonomous ordinary differential equations. Consider the equation

$$y'' = F(y, y'),$$

then we can rewrite this as a system by identifying $x_1 = y$ and $x_2 = y'$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ F(x_1, x_2) \end{pmatrix}.$$

If the system has stationary solutions, they satisfy $x_2 = 0$ and $F(x_1, 0) = 0$. Assume a stationary solution $(x_1, x_2) = (X_0, 0)$. Consider perturbation around it of the form $x_1 = X_0 + \xi$, $x_2 = \eta$, where $\|(\xi, \eta)\|$ is small. Then after linearisation

$$F(x_1, x_2) = a\xi + b\eta + \dots, \quad a = \frac{\partial}{\partial x}F(X_0, 0), \quad b = \frac{\partial}{\partial y}F(X_0, 0),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \eta \\ a\xi + b\eta \end{pmatrix} + \dots = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \dots$$

The matrix has (possibly complex) eigenvalues

$$\lambda_{1,2} = \tfrac{1}{2}b \pm \sqrt{a + \tfrac{1}{4}b^2}.$$

The solutions of the linearised system are typically a linear combination of $\mathrm{e}^{\lambda_1 t}$ and $\mathrm{e}^{\lambda_2 t}$. Depending on the signs of $\lambda_{1,2}$, this results in local behaviour in the phase plane of ellipses (neutrally stable), converging spirals (stable) or diverging spirals (unstable).

Van der Pol's transformation

An interesting class of problems is the nonlinear oscillator

$$y'' + k^2 y + \varepsilon y' g(y, y') = 0.$$

With $g(y, y') = y^2 - 1$ is the Van der Pol equation a famous example. After transformation $t := kt$ and $x_1 = y$, $x_2 = y'$ we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon x_2\, g(x_1, x_2). \end{aligned}$$

Considerable progress can be made if we write the solution in polar coordinates of the phase plane:

$$x_1 = r \sin\varphi, \quad x_2 = r \cos\varphi.$$

This leads to

$$\begin{aligned} \dot{r}\, \sin\varphi + r\, \dot{\varphi}\, \cos\varphi &= r \cos\varphi \\ \dot{r}\, \cos\varphi - r\, \dot{\varphi}\, \sin\varphi &= -r \sin\varphi - \varepsilon r \cos\varphi\, \tilde{g}(r, \varphi). \end{aligned}$$

After eliminating r and φ we have

$$\dot{r} = -\varepsilon r \cos^2\varphi\, \tilde{g}(r,\varphi)$$
$$\dot{\varphi} = 1 + \varepsilon \sin\varphi \cos\varphi\, \tilde{g}(r,\varphi).$$

Since $\varepsilon r \cos^2\varphi \geqslant 0$, the growth ($\dot{r} > 0$) or decay ($\dot{r} < 0$) of the solution depends entirely on the sign of $\tilde{g}$. A consequence is that if $\tilde{g}$ is positive for large r and negative for small r (like the Van der Pol equation), the expanding and contracting phase plots, not being able to cross each other, have to result in (at least) one closed contour (a so-called limit cycle), *i.e.* a periodic solution.

9.2 Newton's equation

An interesting equation that we encounter rather often is Newton's equation

$$y'' + V'(y) = 0, \quad y(0) = y_0, \quad y'(0) = y_1,$$

where V (in mechanical context a potential) is a sufficiently smooth given function of y. The interesting aspect is that the equation does not depend on y' and therefore can be integrated to

$$\tfrac{1}{2}(y')^2 + V(y) = E = \tfrac{1}{2}y_1^2 + V(y_0),$$

with integration constant E. In mechanical context this relation amounts to conservation of total energy E, being the sum of kinetic energy $\frac{1}{2}(y')^2$ and potential energy $V(y)$.

Note that this relation between y and y' is sufficient to construct phase plots for various values of E. For those values of E, where these phase plots correspond to closed curves, we know in advance that the corresponding solutions are periodic, which is important information.

We can eliminate y' and obtain

$$y' = \pm\sqrt{2}\sqrt{E - V(y)}.$$

Furthermore, we can even determine y implicitly formally

$$\int_{y_0}^{y} \frac{1}{\sqrt{E - V(s)}}\,\mathrm{d}s = \pm\sqrt{2}\,t$$

and with some luck we can integrate this integral explicitly. Note that a full integration may depend on the value of E.

A simple but important example is

$$y'' + k^2 y = 0$$

with ellipses in the phase plane described by

$$\tfrac{1}{2}(y')^2 + \tfrac{1}{2}k^2y^2 = E = \tfrac{1}{2}y_1^2 + \tfrac{1}{2}k^2y_0^2$$

leading to

$$\int_{y_0}^{y} \frac{1}{\sqrt{E - \frac{1}{2}k^2s^2}}\,\mathrm{d}s = \frac{\sqrt{2}}{k}\arcsin\left(\frac{ks}{\sqrt{2E}}\right)\bigg|_{y_0}^{y} = \pm\sqrt{2}\,t.$$

The integral describes one period ($+$ for one half and $-$ for another half), that can be extended. Hence we obtain the expected $y = y_0 \cos(kt) + y_1 k^{-1} \sin(kt)$.

Another, less trivial example is

$$y'' + y - y^3 = 0$$

with

$$\tfrac{1}{2}(y')^2 + \tfrac{1}{2}y^2 - \tfrac{1}{4}y^4 = E.$$

Elementary analysis (check when the zero's of $E - x + x^2$ are positive and real) reveals that this relation yields in the phase plane closed curves around the origin if $0 < E \leqslant \frac{1}{4}$. Hence, there are periodic solutions for those values of E.

9.3 Normal vectors of level surfaces

A convenient way to describe a smooth surface $\mathcal{S}$ is by means of a suitable smooth function $S(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y, z)$, chosen such that the level surface $S(\boldsymbol{x}) = 0$ coincides with $\mathcal{S}$. So $S(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} \in \mathcal{S}$. (Example: $x^2 + y^2 + z^2 - R^2 = 0$ for a sphere; $z - h(x, y) = 0$ for a landscape.)

Then for closely located points $\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{h} \in \mathcal{S}$ we have

$$S(\boldsymbol{x} + \boldsymbol{h}) = S(\boldsymbol{x}) + \boldsymbol{h} \cdot \nabla S(\boldsymbol{x}) + O(\boldsymbol{h}^2) \simeq \boldsymbol{h} \cdot \nabla S(\boldsymbol{x}) = 0.$$

Since $\boldsymbol{h}$ is (for $\boldsymbol{h} \to 0$) a tangent vector of $\mathcal{S}$, it follows that ∇S at $S = 0$ is a normal of $\mathcal{S}$ (provided $\nabla S \neq 0$). We write $\boldsymbol{n} \sim \nabla S\big|_{S=0}$.

9.4 Trigonometric relations

The real or imaginary parts of the binomial series $(\mathrm{e}^{\mathrm{i}x} \pm \mathrm{e}^{-\mathrm{i}x})^n = \sum_{k=0}^{n} (\pm)^k \binom{n}{k} \mathrm{e}^{\mathrm{i}(n-2k)x}$ easily yield trigonometric relations, useful for recognising resonance terms:

$$
\begin{aligned}
\sin^2 x &= \tfrac{1}{2}(1 - \cos 2x), \\
\sin x \cos x &= \tfrac{1}{2}\sin 2x, \\
\cos^2 x &= \tfrac{1}{2}(1 + \cos 2x),
\end{aligned}
\qquad
\begin{aligned}
\sin^3 x &= \tfrac{1}{4}(3\sin x - \sin 3x), \\
\sin^2 x \cos x &= \tfrac{1}{4}(\cos x - \cos 3x), \\
\sin x \cos^2 x &= \tfrac{1}{4}(\sin x + \sin 3x), \\
\cos^3 x &= \tfrac{1}{4}(3\cos x + \cos 3x),
\end{aligned}
$$

$$
\begin{aligned}
\sin^4 x &= \tfrac{1}{8}(3 - 4\cos 2x + \cos 4x), \\
\sin^3 x \cos x &= \tfrac{1}{8}(2\sin 2x - \sin 4x), \\
\sin^2 x \cos^2 x &= \tfrac{1}{8}(1 - \cos 4x), \\
\sin x \cos^3 x &= \tfrac{1}{8}(2\sin 2x + \sin 4x), \\
\cos^4 x &= \tfrac{1}{8}(3 + 4\cos 2x + \cos 4x),
\end{aligned}
\qquad
\begin{aligned}
\sin^5 x &= \tfrac{1}{16}(10\sin x - 5\sin 3x + \sin 5x), \\
\sin^4 x \cos x &= \tfrac{1}{16}(2\cos x - 3\cos 3x + \cos 5x), \\
\sin^3 x \cos^2 x &= \tfrac{1}{16}(2\sin x + \sin 3x - \sin 5x), \\
\sin^2 x \cos^3 x &= \tfrac{1}{16}(2\cos x - \cos 3x - \cos 5x), \\
\sin x \cos^4 x &= \tfrac{1}{16}(2\sin x + 3\sin 3x + \sin 5x), \\
\cos^5 x &= \tfrac{1}{16}(10\cos x + 5\cos 3x + \cos 5x).
\end{aligned}
$$

Another way to understand these relations is as (finite) Fourier series expansions of the 2π-periodic functions at the left hand side. In particular

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,\mathrm{d}x, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

Chapter 10

Special Functions

The following functions are important in applied analysis, especially in physical applications. They play a role in, and have been subject of asymptotic analysis.

10.1 Bessel Functions

The Bessel equation

$$y'' + \frac{1}{z}y' + \Big(1 - \frac{n^2}{z^2}\Big)y = 0,$$

and therefore its solutions the Bessel Functions, appears naturally when the Laplace operator is rewritten in polar coordinates[1]. We have the following standard forms

$$\begin{aligned}
J_n(z) &= \frac{1}{\pi}\int_0^{\pi} \cos(z\sin t - nt)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}z\sin t - \mathrm{i}nt}\,\mathrm{d}t,\\
Y_n(z) &= \frac{1}{\pi}\int_0^{\pi} \sin(z\sin t - nt)\,\mathrm{d}t - \frac{1}{\pi}\int_0^{\infty} \left[\mathrm{e}^{nt} + (-1)^n\,\mathrm{e}^{-nt}\right]\mathrm{e}^{-z\sinh t}\,\mathrm{d}t,\\
H_n^{(1)}(z) &= J_n(z) + \mathrm{i}Y_n(z) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{+\infty+\pi\mathrm{i}} \mathrm{e}^{z\sinh t - nt}\,\mathrm{d}t,\\
H_n^{(2)}(z) &= J_n(z) - \mathrm{i}Y_n(z) = -\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{+\infty-\pi\mathrm{i}} \mathrm{e}^{z\sinh t - nt}\,\mathrm{d}t.
\end{aligned}$$

J_n is called the ordinary n-th order Bessel Function of the 1st kind; Y_n is called the n-th order Bessel Function of the 2nd kind or Neumann Function; $H_n^{(1)}$, $H_n^{(2)}$ are called the n-th order Hankel Functions of the 1st and 2nd kind or Bessel Functions of the 3d kind.

[1] In 2D we have $\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta}$. General solutions of $\nabla^2\phi = 0$ are found by separation of variables $\phi = f(r)g(\theta)$, leading to $\phi = \sum f_n(r)g_n(\theta)$ where $r^2 f_n'' + r f_n' + (r^2 - n^2) f_n = 0$ and $g_n'' + n^2 g_n = 0$, for $n \in \mathbb{Z}$.

10.2 Gamma Function

For $\mathrm{Re}(z) > 0$ is $\Gamma(z)$, known as the Gamma Function, and the related $z!$ defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \qquad z! = \Gamma(z+1).$$

The identities $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1) = 1$ yield indeed $\Gamma(n+1) = n(n-1)\cdots 1$. By pulling back like $\Gamma(z) = \Gamma(z+1)/z = \Gamma(z+2)/z(z+1) = \ldots$, we extend the definition to all $z \in \mathbb{C}$. With Euler's Reflection Formula

$$\Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi$$

one may derive that Γ is analytic everywhere except for $z = -n$, $n = 0, 1, 2, \ldots$ where it has simple poles with residue $(-1)^n/n!$. For the asymptotic behaviour of $\Gamma(z)$, see example 8.5.

The logarithmic derivative is known as the Digamma Function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \; \psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \cdots \quad (z \to \infty \;\text{ in }\; |\arg(z)| \leqslant \pi - \delta)$$

For $z \in \mathbb{N}$ and Euler's Constant $\gamma = 0.5772156649\ldots$ we have the result for the harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = \psi(n+1) + \gamma \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \cdots \quad (n \to \infty)$$

The Incomplete Gamma Functions are defined by the integrals

$$\Gamma(a, z) = \int_z^\infty t^{a-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \qquad \gamma(a, z) = \int_0^z t^{a-1}\,\mathrm{e}^{-t}\,\mathrm{d}t \quad (\mathrm{Re}(a) > 0).$$

Unless indicated otherwise, principal values are assumed with a branch cut along the negative real axis and the integration contours not crossing the negative real axis. Note that $\Gamma(0, z) = \mathrm{E}_1(z)$, $\Gamma(1, z) = \mathrm{e}^{-z}$, $\Gamma(\frac{1}{2}, z^2) = \sqrt{\pi}\,\mathrm{erfc}(z)$. Asymptotically for $z \to \infty$ and a fixed we have

$$\Gamma(a, z) = z^{a-1}\,\mathrm{e}^{-z}\left(\sum_{k=0}^{n-1} \frac{u_k}{z^k} + O(z^{-n})\right), \quad \text{in} \quad |\arg(z)| \leqslant \tfrac{3}{2}\pi - \delta.$$

where $u_0 = 1$, $u_k = (a-1)(a-2)\cdots(a-k)$.

A related function is $B(x, y)$, known as the Beta Function, and defined for $\mathrm{Re}\, x > 0$, $\mathrm{Re}\, y > 0$ by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,\mathrm{d}t.$$

Example 10.1 Let $\alpha \in \mathbb{R}$ and $\beta, z \in \mathbb{C}$ with $\alpha > 0$ and $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}\, z > 0$, and a principal value power function. Then

$$\int_0^\infty t^{\beta-1}\,\mathrm{e}^{-zt^\alpha}\,\mathrm{d}t = \alpha^{-1} z^{-\beta/\alpha}\,\Gamma(\beta/\alpha).$$

Example 10.2 Let $z, \alpha \in \mathbb{C}$. Then

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = 1 + \alpha z + \tfrac{1}{2}\alpha(\alpha-1)z^2 + \cdots, \quad \binom{\alpha}{n} = \frac{\alpha!}{n!(\alpha-n)!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

where the series is finite if $\alpha \in \mathbb{N} \cup \{0\}$. Otherwise, it converges absolutely for $|z| < 1$.

10.3 Dilogarithm and Exponential Integral

The Dilogarithm and the Exponential Integral are complex functions with much in common with the complex logarithm.

The Dilogarithm, defined by

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\ln(1-t)}{t}\,\mathrm{d}t = \int_0^{\infty} \frac{zt}{\mathrm{e}^t - z}\,\mathrm{d}t$$

has a branch point[2] in $z = 0$ and a branch cut along the negative real axis. Special values:

$$\mathrm{Li}_2(1) = \tfrac{1}{6}\pi, \qquad \mathrm{Li}_2(-1) = -\tfrac{1}{12}\pi.$$

Note that another definition is known, given by $\mathrm{dilog}(z) = \mathrm{Li}_2(1-z)$.

We define the Exponential Integral

$$\mathrm{E}_1(z) = \int_z^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t = -\gamma - \log(z) - \sum_{k=1}^{\infty} \frac{(-z)^k}{kk!} = -\gamma - \log(z) + \int_0^z \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t,$$

where $\gamma = 0.5772156649\ldots$ is Euler's Constant, and with a branch point in $z = 0$ and a branch cut along the negative real axis. Note that another definition is known, given by $\mathrm{Ei}(z) = -\mathrm{E}_1(-z)$.

10.4 Error Function

The Error Function is defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

The related Complementary Error Function is given by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t = 1 - \mathrm{erf}(z).$$

Special values:

$$\mathrm{erf}(\infty) = \mathrm{erfc}(0) = 1.$$

For large real x and any fixed $N \in \mathbb{N}$ we have asymptotically

$$\mathrm{erfc}(x) = \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}} \left[1 + \sum_{n=1}^{N} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} + O(x^{-2N-1}) \right] \quad (x \to \infty).$$

[2]For other branches than the principal branch, there is a second branch point in $z = 1$.

Chapter 11

Units, Dimensions, Dimensionless Numbers

Basic units			
Name	Symbol	Physical quantity	Unit
meter	m	length	m
kilogram	kg	mass	kg
second	s	time	s
ampere	A	electric current	A
kelvin	K	temperature	K
candela	cd	luminous intensity	cd
mole	mol	amount of substance	1
hertz	Hz	frequency	$1/s$
newton	N	force, weight	$kg\,m/s^2$
pascal	Pa	pressure, stress	N/m^2
joule	J	energy, work, heat	$N\,m$
watt	W	power	J/s
radian	rad	planar angle	1
steradian	sr	solid angle	1
coulomb	C	electric charge	$A\,s$
volt	V	electric potential	$kg\,m^2/s^3A$
ohm	Ω	electric resistance	$kg\,m^2/s^3A^2$
siemens	S	electric conductance	$1/\Omega$
lumen	lm	luminous flux	$cd\,sr$
lux	lx	illuminance	lm/m^2

Basic variables			
Quantity	Relation	Unit	Dimensions
stress	force/area	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
pressure	force/area	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
Young's modulus	stress/strain	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
Lamé parameters λ and μ	stress/strain	$\mathrm{N/m^2 = Pa}$	$\mathrm{kg\,m^{-1}\,s^{-2}}$
strain	displacement/length	1	1
Poisson's ratio	transverse strain/axial strain	1	1
density	mass/volume	$\mathrm{kg/m^3}$	$\mathrm{kg\,m^{-3}}$
velocity	length/time	$\mathrm{m/s}$	$\mathrm{m\,s^{-1}}$
acceleration	velocity/time	$\mathrm{m/s^2}$	$\mathrm{m\,s^{-2}}$
(linear) momentum	mass $\times$ velocity	$\mathrm{kg\,m/s}$	$\mathrm{kg\,m\,s^{-1}}$
force	momentum/time	N	$\mathrm{kg\,m\,s^{-2}}$
impulse	force $\times$ time	$\mathrm{N\,s}$	$\mathrm{kg\,m\,s^{-1}}$
angular momentum	distance $\times$ mass $\times$ velocity	$\mathrm{kg\,m^2/s}$	$\mathrm{kg\,m^2\,s^{-1}}$
moment (of a force)	distance $\times$ force	$\mathrm{N\,m}$	$\mathrm{kg\,m^2\,s^{-2}}$
work	force $\times$ distance	$\mathrm{N\,m = J}$	$\mathrm{kg\,m^2\,s^{-2}}$
heat	work	J	$\mathrm{kg\,m^2\,s^{-2}}$
energy	work	$\mathrm{N\,m = J}$	$\mathrm{kg\,m^2\,s^{-2}}$
power	work/time, energy/time	$\mathrm{J/s = W}$	$\mathrm{kg\,m^2\,s^{-3}}$
heat flux	heat rate/area	$\mathrm{W/m^2}$	$\mathrm{kg\,s^{-3}}$
heat capacity	heat change/temperature change	$\mathrm{J/K}$	$\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}$
specific heat capacity	heat capacity/unit mass	$\mathrm{J/K\,kg}$	$\mathrm{m^2\,s^{-2}\,K^{-1}}$
thermal conductivity	heat flux/temperature gradient	$\mathrm{W/m\,K}$	$\mathrm{kg\,m\,s^{-3}\,K^{-1}}$
dynamic viscosity	shear stress/velocity gradient	$\mathrm{kg/m\,s}$	$\mathrm{kg\,m^{-1}\,s^{-1}}$
kinematic viscosity	dynamic viscosity/density	$\mathrm{m^2/s}$	$\mathrm{m^2\,s^{-1}}$
surface tension	force/length	$\mathrm{N/m}$	$\mathrm{kg\,s^{-2}}$

Dimensionless numbers			
Name	Symbol	Definition	Description
Archimedes	Ar	$g\Delta\rho L^3/\rho\nu^2$	particles, drops or bubbles
Arrhenius	Arr	E/RT	chemical reactions
Biot	Bi	hL/κ	heat transfer at surface of body
Biot	Bi	$h_D L/D$	mass transfer
Bodenstein	Bo	VL/D_{ax}	mass transfer with axial dispersion
Bond	Bo	$\rho g L^2/\sigma$	gravity against surface tension
Capillary	Ca	$\mu V/\sigma$	viscous forces against surface tension
Dean	De	$(VL/\nu)(L/2r)^{1/2}$	flow in curved channels
Eckert	Ec	$V^2/C_P\Delta T$	kinetic energy against enthalpy difference
Euler	Eu	$\Delta p/\rho V^2$	pressure resistance
Fourier	Fo	$\alpha t/L^2$	heat conduction
Fourier	Fo	Dt/L^2	diffusion
Froude	Fr	$V/(gL)^{1/2}$	gravity waves
Galileo	Ga	$gL^3\rho^2/\mu^2$	gravity against viscous forces
Grashof	Gr	$\beta\Delta T g L^3/\nu^3$	natural convection
Helmholtz	He	$\omega L/c = kL$	acoustic wave number
Kapitza	Ka	$g\mu^4/\rho\sigma^3$	film flow
Knudsen	Kn	λ/L	low density flow
Lewis	Le	α/D	combined heat and mass transfer
Mach	M	V/c	compressible flow
Nusselt	Nu	hL/κ	convective heat transfer
Ohnesorge	Oh	$\mu/(\rho L\sigma)^{1/2}$	viscous forces, inertia and surface tension
Péclet	Pe	VL/α	forced convection heat transfer
Péclet	Pe	VL/D	forced convection mass transfer
Prandtl	Pr	$\nu/\alpha = C_P\mu/\kappa$	convective heat transfer
Rayleigh	Ra	$\beta\Delta T g L^3/\alpha\nu$	natural convection heat transfer
Reynolds	Re	$\rho VL/\mu$	viscous forces against inertia
Schmidt	Sc	ν/D	convective mass transfer
Sherwood	Sh	$h_D L/D$	convective mass transfer
Stanton	St	$h/\rho C_P V$	forced convection heat transfer
Stanton	St	h_D/V	forced convection mass transfer
Stokes	S	ν/fL^2	viscous damping in unsteady flow
Strouhal	Sr	fL/V	hydrodynamic wave number
Weber	We	$\rho V^2 L/\sigma$	film flow, bubble formation, droplet breakup

Nomenclature		
Symbol	Description	Units
c	sound speed	m/s
C_P	specific heat	J/kg K
D	diffusion coefficient	m^2/s
Dax	axial dispersion coefficient	m^2/s
E	activation energy	J/mol
f	frequency	1/s
g	gravitational acceleration	m/s^2
h	heat transfer coefficient	W/m^2 K
h_D	mass transfer coefficient	m/s
k	wave number = ω/c	1/m
L	length	m
$p, \Delta p$	pressure	Pa
R	universal gas constant	J/mol K
r	radius of curvature	m
$T, \Delta T$	temperature	K
t	time	s
V	velocity	m/s
$\alpha = \kappa/\rho C_P$	thermal diffusivity	m^2/s
β	coef. of thermal expansion	K^{-1}
κ	thermal conductivity	W/m K
λ	molecular mean free path	m
μ	dynamic viscosity	Pa s
$\nu = \mu/\rho$	kinematic viscosity	m^2/s
$\rho, \Delta\rho$	density	kg/m^3
σ	surface tension	N/m
ω	circular frequency = $2\pi f$	1/s

Quotes

1. *The little things are infinitely the most important.* (Sherlock Holmes.)
2. *Entia non sunt multiplicanda praeter necessitatem = Entities should not be multiplied beyond necessity* $\approx$ *Other things being equal, simpler explanations are generally better than more complex ones.* (W. Ockham.)
3. *Formulas are wiser than man.* (J. de Graaf.)
4. *Nothing is as practical as a good theory.* (J.R. Oppenheimer.)
5. *An approximate answer to the right question is worth a great deal more than a precise answer to the wrong question.* (J. Tukey.)
6. *An exact solution of an approximate model is not better than an approximate solution of an exact model.* (section 2.)
7. *Never make a calculation until you know the answer: make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every puzzle.* (J.A. Wheeler.)
8. *The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.* (G.H. Hardy.)
9. *Divide each difficulty into as many parts as is feasible and necessary to resolve it.* (R. Descartes.)
10. *You make experiments and I make theories. Do you know the difference? A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it.* (A. Einstein.)
11. *It is the theory which decides what we can observe.* (A. Einstein.)
12. *As far as the laws of mathematics refer to reality, they are not certain, as far as they are certain, they do not refer to reality.* (A. Einstein.)
13. *Science is nothing without generalisations. Detached and ill-assorted facts are only raw material, and in the absence of a theoretical solvent, have but little nutritive value.* (Lord Rayleigh)
14. *We need vigour, not rigour!* (anonym.)
15. *It is the nature of all greatness not to be exact.* (E. Burke.)
16. *The capacity to learn is a gift; The ability to learn is a skill; The willingness to learn is a choice.* (F. Herbert.)

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